Detection of dark matter axions 2/3

from classical microwaves to quantised photons

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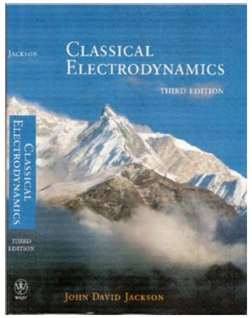
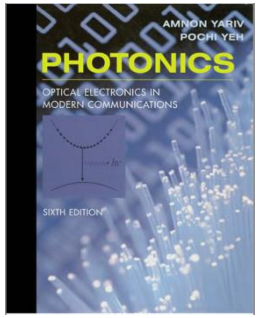
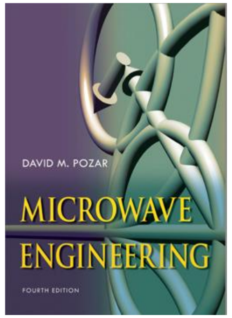
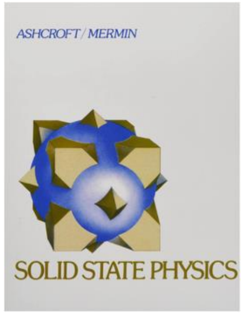
Outline of the lecture courses and textbooks

- Part 1: overview on axion searches
  - Axion vs WIMPs
  - Various experiments
  - Non-DM axions
- Part 2: classical detection scheme
  - Boundary conditions
  - Microwave resonators
  - Analog and digital system
  - Data processing and noise

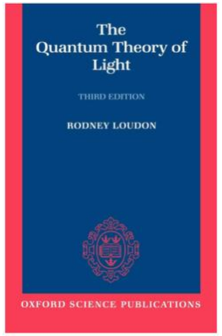


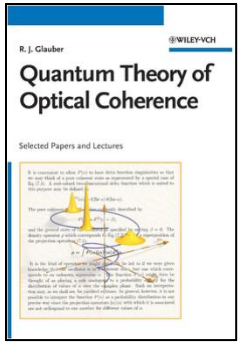




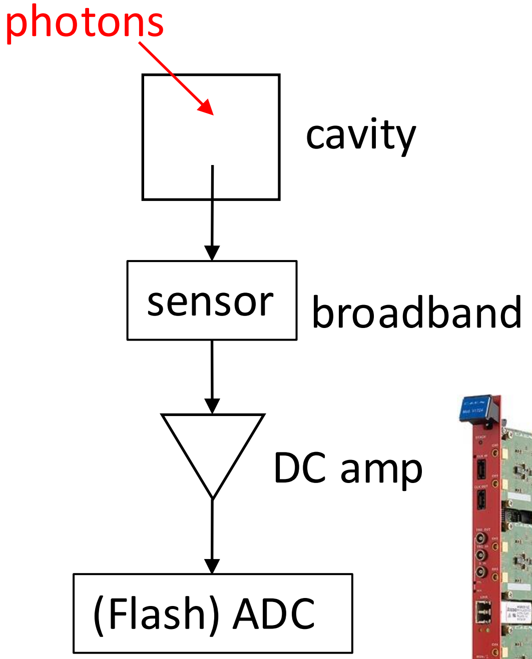


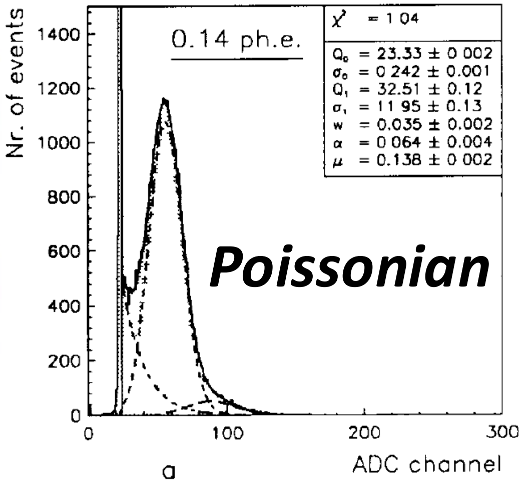
- Part 3: quantum detection scheme
  - Quantum coherent states
  - Glauber's theorem
  - Thermal noise and Standard Quantum Limit
  - Squeezing and photon counting



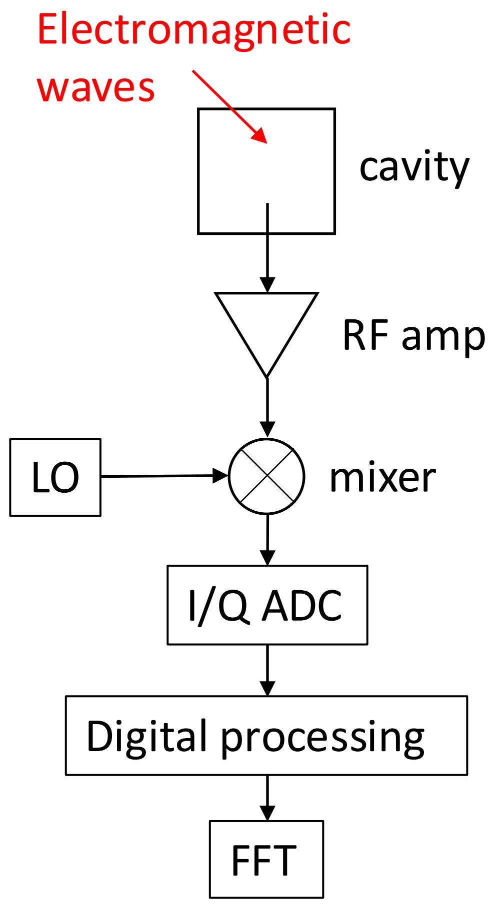


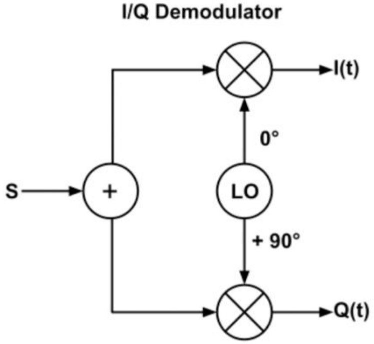
Photon (energy) detection vs wave detection



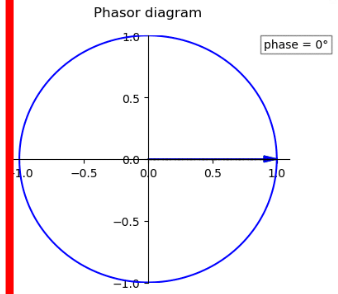


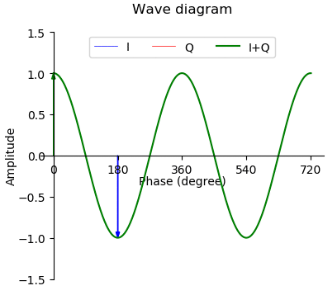
This lecture

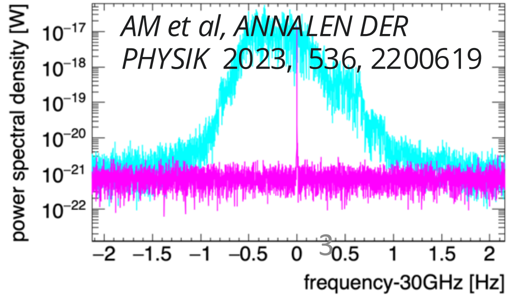




From wikipedia







$$P(t) = n \times \hbar\omega \propto V_{ADC}(t)$$

$$\Delta\phi\Delta n > 1$$

$$RF(t) = I(t)\cos(\omega t) + Q(t)\sin(\omega t)$$

$$\rightarrow P(\omega) = \tilde{I}^2(\omega) + \tilde{Q}^2(\omega)$$

Part 2: classical detection scheme

- Boundary conditions
  - Normal conductors
  - Superconductors
  - Dielectric materials: insulator and semi-conductors
- Microwave resonators
  - Waveguide and transmission line
  - Resonant cavity
  - Fabry-Pérot resonator
- Analog and digital system
  - Amplifier, circulator, mixer & analogue down-conversion
  - I/Q sampling & digital down-conversion
- Data processing and noise
  - FFT: coherent and incoherent integral
  - Narrowband and broadband
  - Thermal noise and standard quantum limit
- Conclusion of part 2

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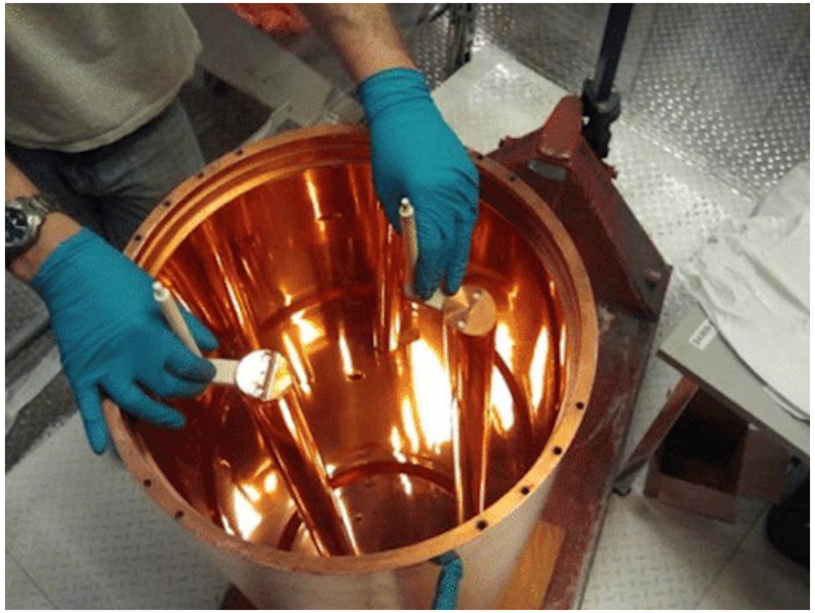
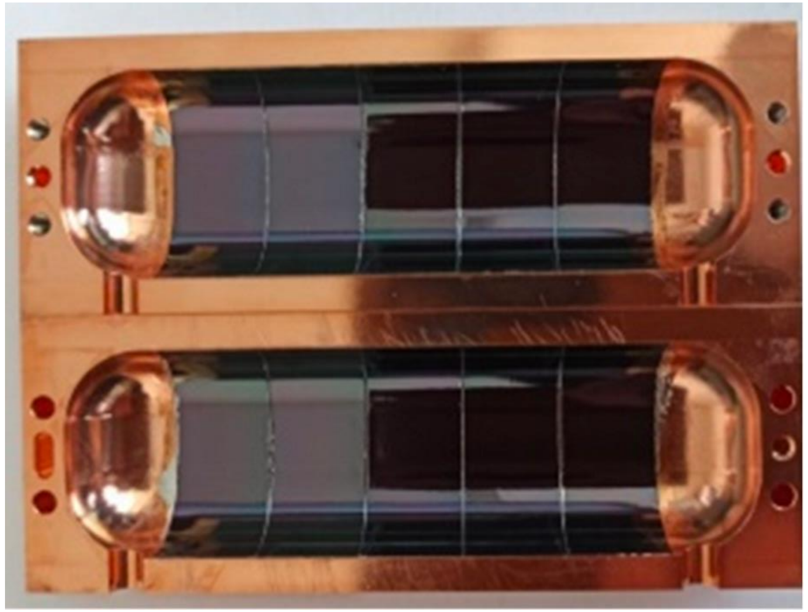
Three types of materials in axion experiments

Normal conducting metal



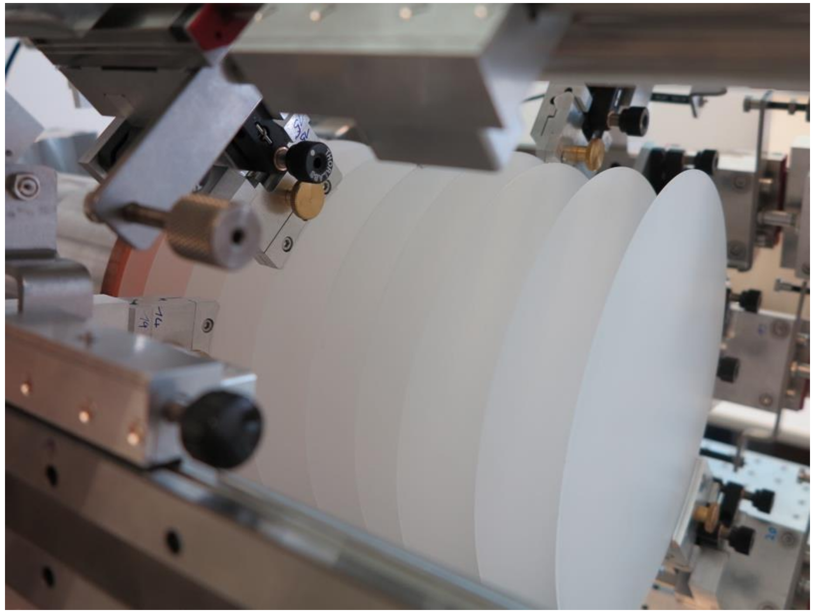
ADMX

Superconducting metal



RADES

Dielectric materials



MADMAX

Our interest: (unloaded) quality factor

Axion signal power

$$P_S \propto Q_0$$

Higher stored U with smaller power dissipation P_c

$$Q_0 = \frac{\omega U}{P_c}$$

Smaller surface resistance R_s
→ high Q & low P_c

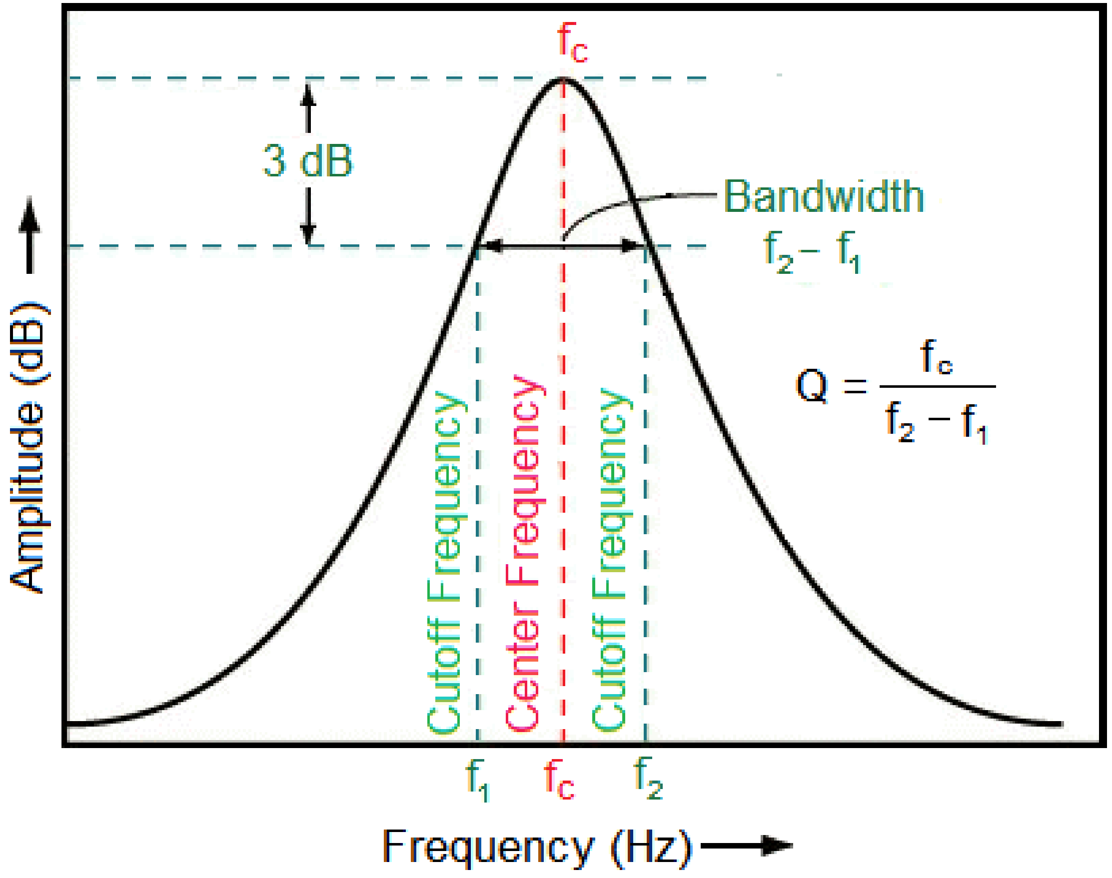
$$Q_0 = \frac{G}{R_s}$$

Experimental observable

Geometrical

From material

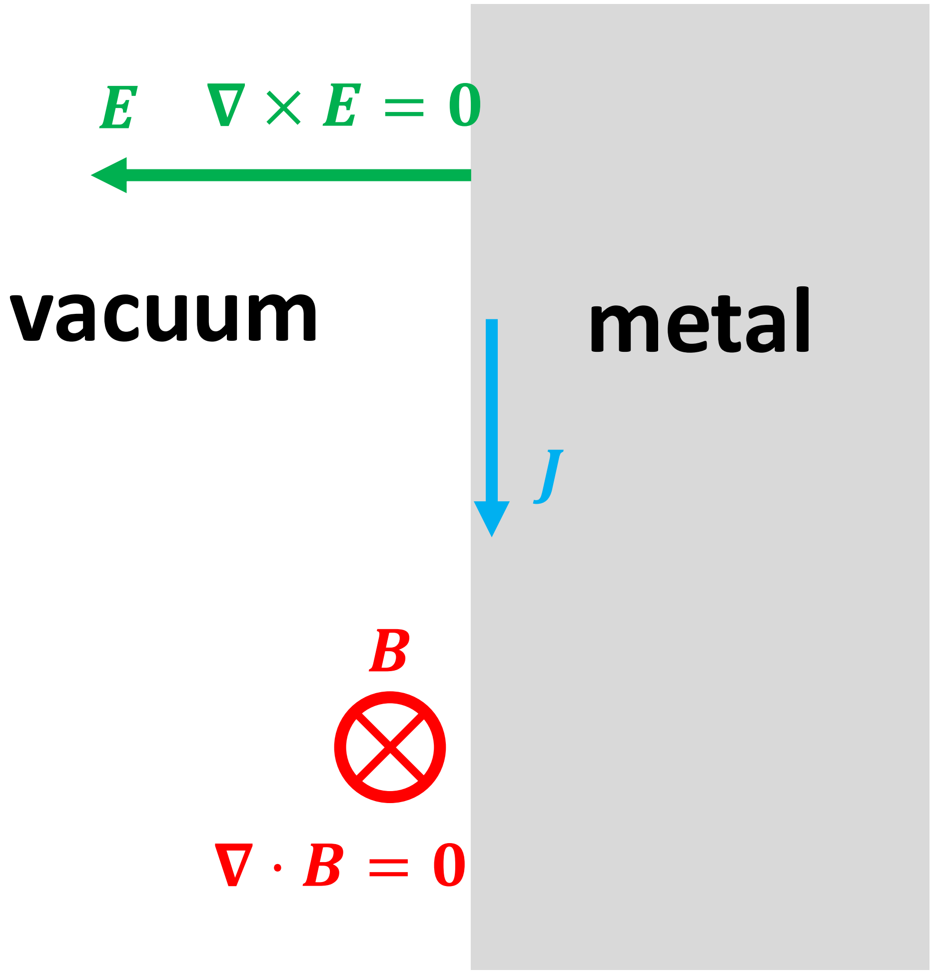
http://lossenderosstudio.com/glossary.php?index=q



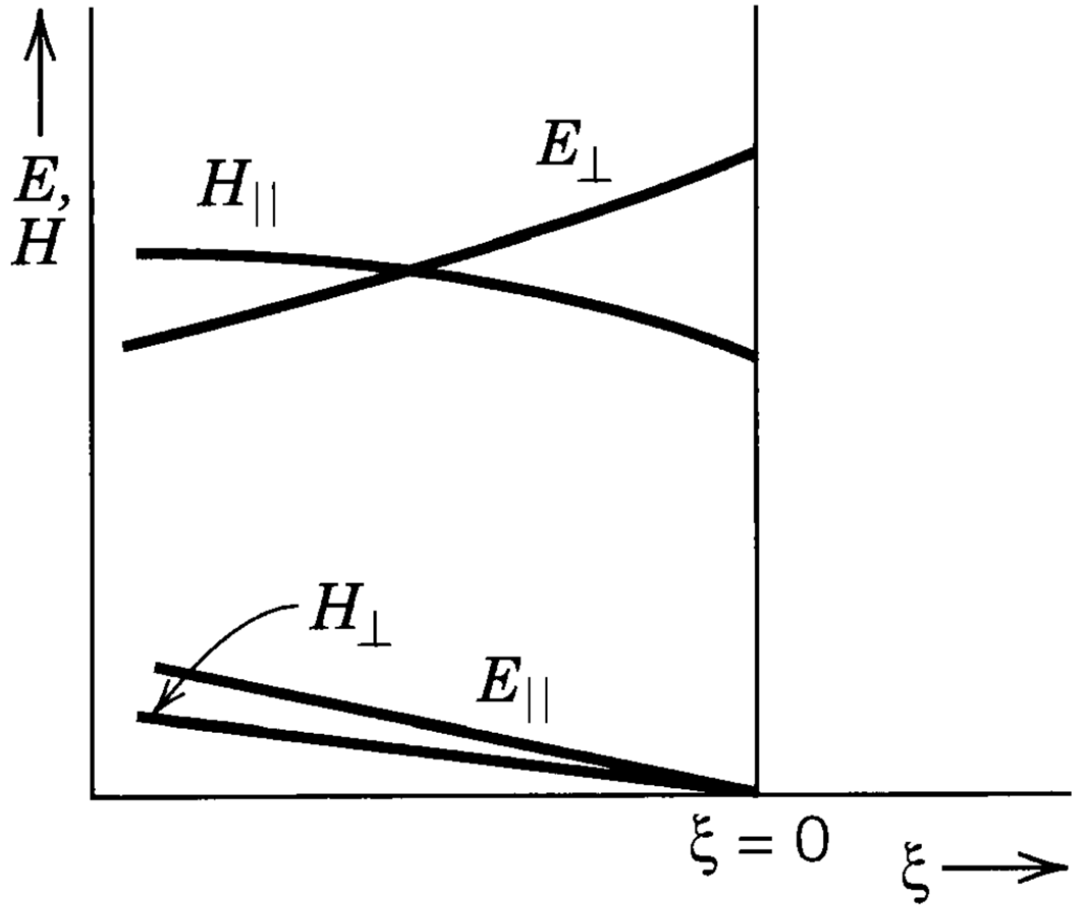
<u>Rule of thumb</u>

Copper's $Q_0 \sim 10^{4-5}$, SC $Q_0 \sim 10^{10-11}$, SC under B $Q_0 \sim 10^6$, dielectrics under B $Q_0 \sim 10^{6-7}$

Vacuum / Perfect Electric Conductor (PEC)

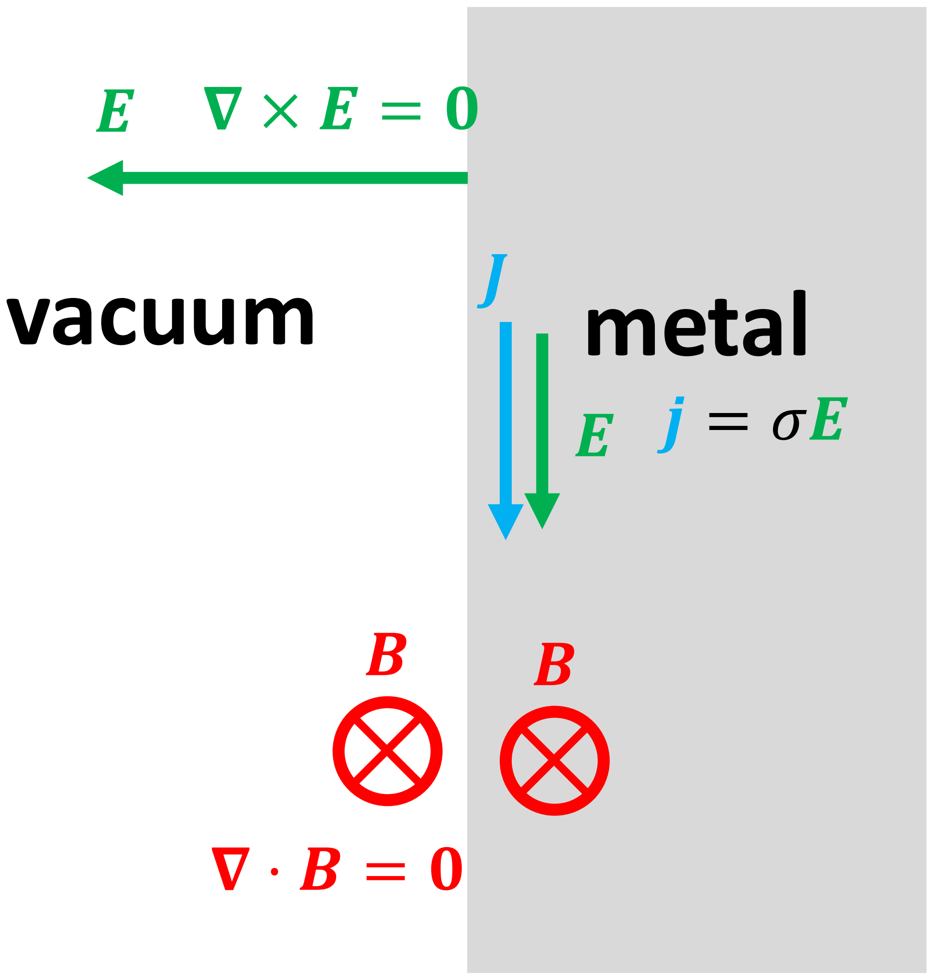


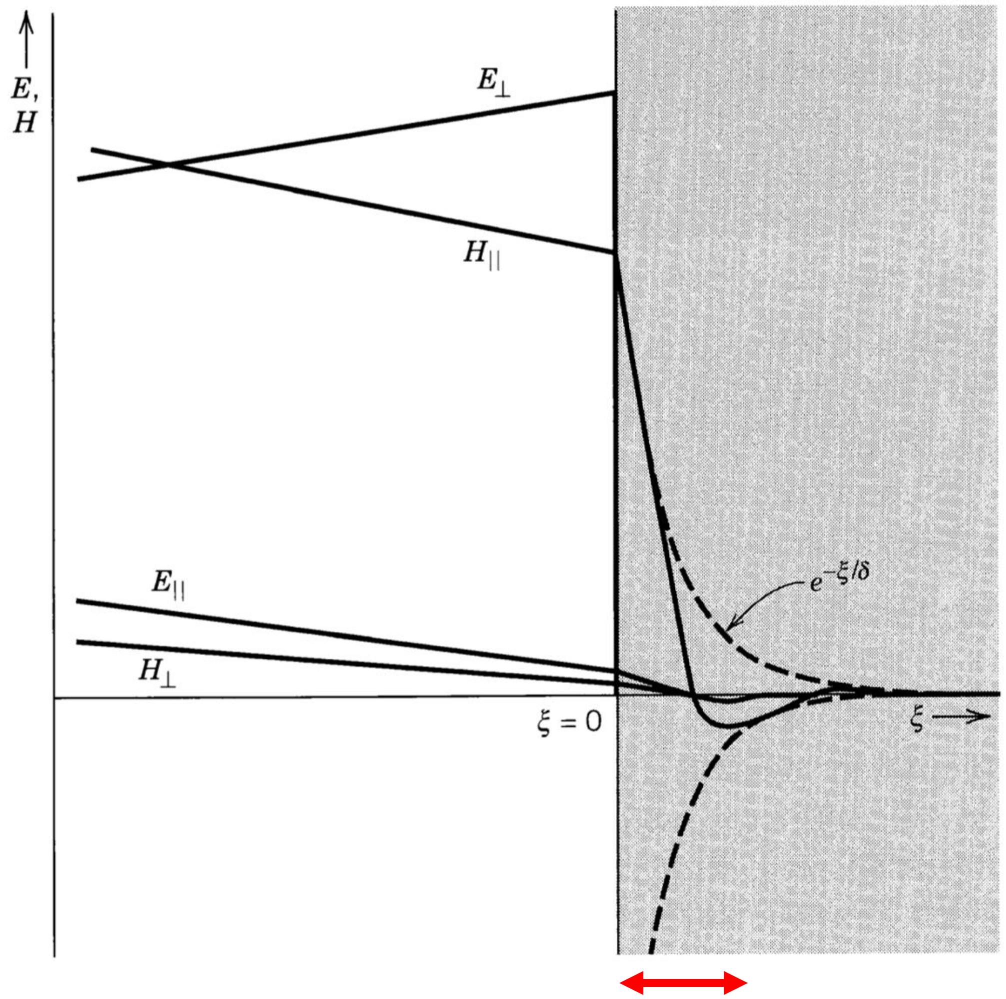
Jackson 3rd edition p. 353



Real normal conducting metal

Jackson 3rd edition p. 355





Skin effect → finite loss

Response of normal conductors against RF

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = \sigma \boldsymbol{E} + \varepsilon\mu \frac{\partial \boldsymbol{E}}{\partial t}$$

$$\boldsymbol{E} = \boldsymbol{E}_0 \exp(i\omega t), \quad \boldsymbol{B} \propto \boldsymbol{B}_0 \exp(i\omega t) \longrightarrow \begin{cases} \nabla \times \boldsymbol{E} = -i\omega \boldsymbol{B} \\ \nabla \times \boldsymbol{B} = i\omega\varepsilon\mu \boldsymbol{E} + \sigma \boldsymbol{E} \end{cases}$$

$$\rightarrow \nabla^2 \boldsymbol{E} + \boxed{\omega^2 \mu\varepsilon \left(1 - \frac{i\sigma}{\omega\varepsilon}\right)} \boldsymbol{E} = 0$$

$= -\gamma^2$:complex propagation constant

$$\gamma = \alpha + i\beta = i\omega\sqrt{\mu\varepsilon}\sqrt{1 - \frac{\sigma}{i\omega\varepsilon}} \sim (1+i)\sqrt{\frac{\omega\mu\sigma}{2}} = \frac{1+i}{\delta_s}$$

Imaginary part = the Lez's law or (eddy current) from Ohm's law not from displacement current

$$\delta_s \equiv \sqrt{\frac{2}{\omega\mu\sigma}}$$ Skin depth

$$R_s = \frac{1}{\sigma\delta} \propto f^{1/2}$$ Surface resistance

→ Normal conductors respond to RF with phase shift but Re and Im parts are the same

→ Very weak diamagnetic effect in normal conductors

Residual Resistivity Ratio (RRR)

$$\frac{1}{\tau} = \frac{1}{\tau_{\mathrm{def}}} + \frac{1}{\tau_{ph}}$$

Phonons get frozen at cold

$$T \downarrow \rightarrow \tau_{ph} \uparrow$$

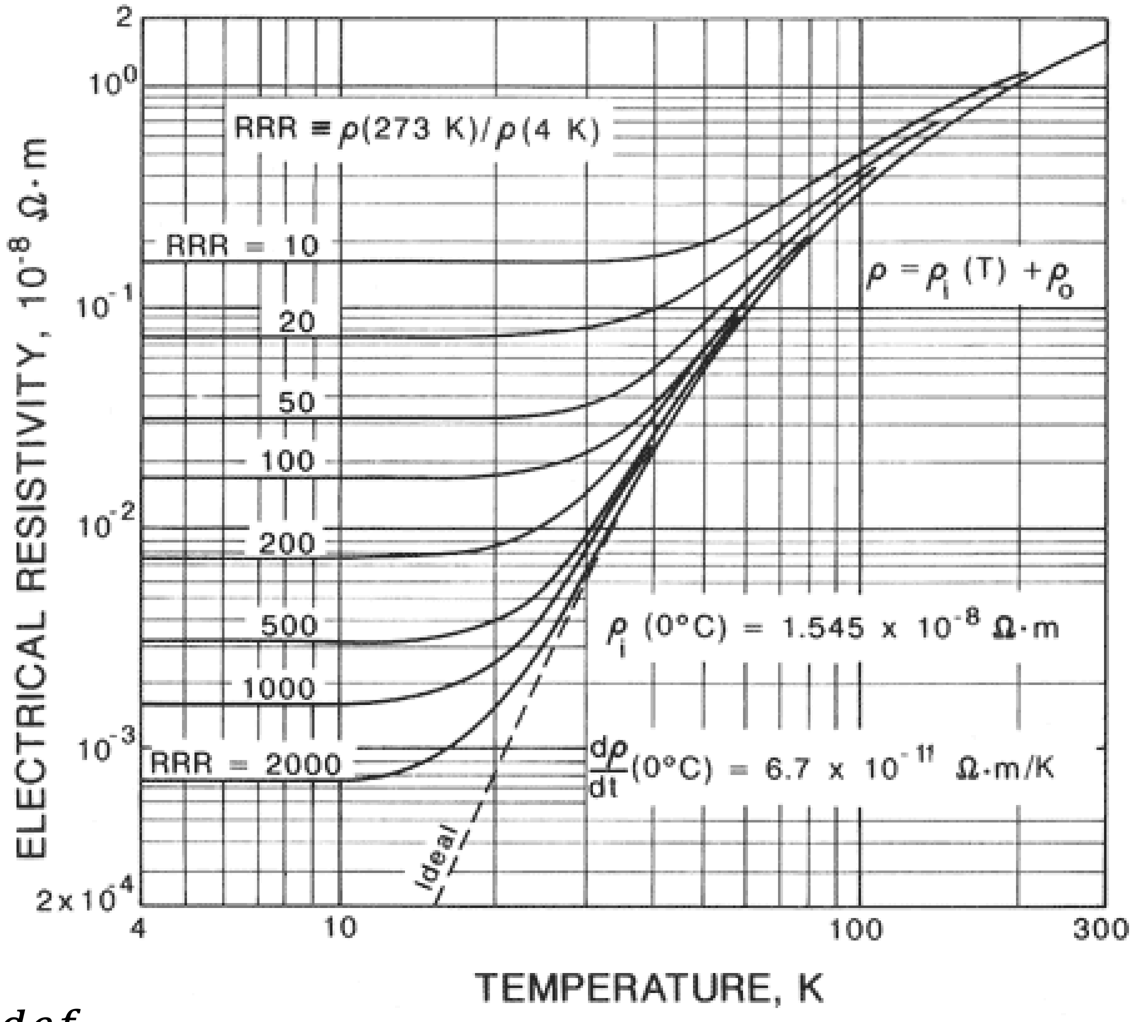
DC resistivity gets better at cold

$$RRR \equiv \frac{\sigma(< 10K)}{\sigma(300K)} = \frac{\rho(300K)}{\rho(< 10K)} > 1$$

Characteristic parameter of purity of metals

$$l \propto RRR$$

Improvement is stuck due to defect scattrring τ_{def}



https://www.copper.org/resources/properties/cryogenic/

Paired electrons can avoid Ohmic loss (τ_{def})

If electrons ***in a distance*** (>39 nm) are bounded, ***local*** (< 0.5 nm) scattering can be avoided

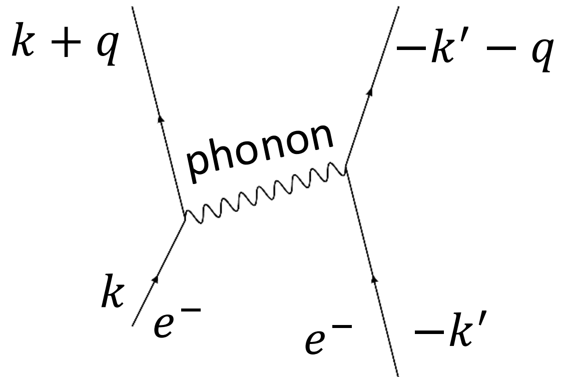
Any small attractive interaction V between electrons can lead to a ***Cooper pair*** coupled with an energy 2Δ, below critical temperature T_c

BCS gap equation (1957)

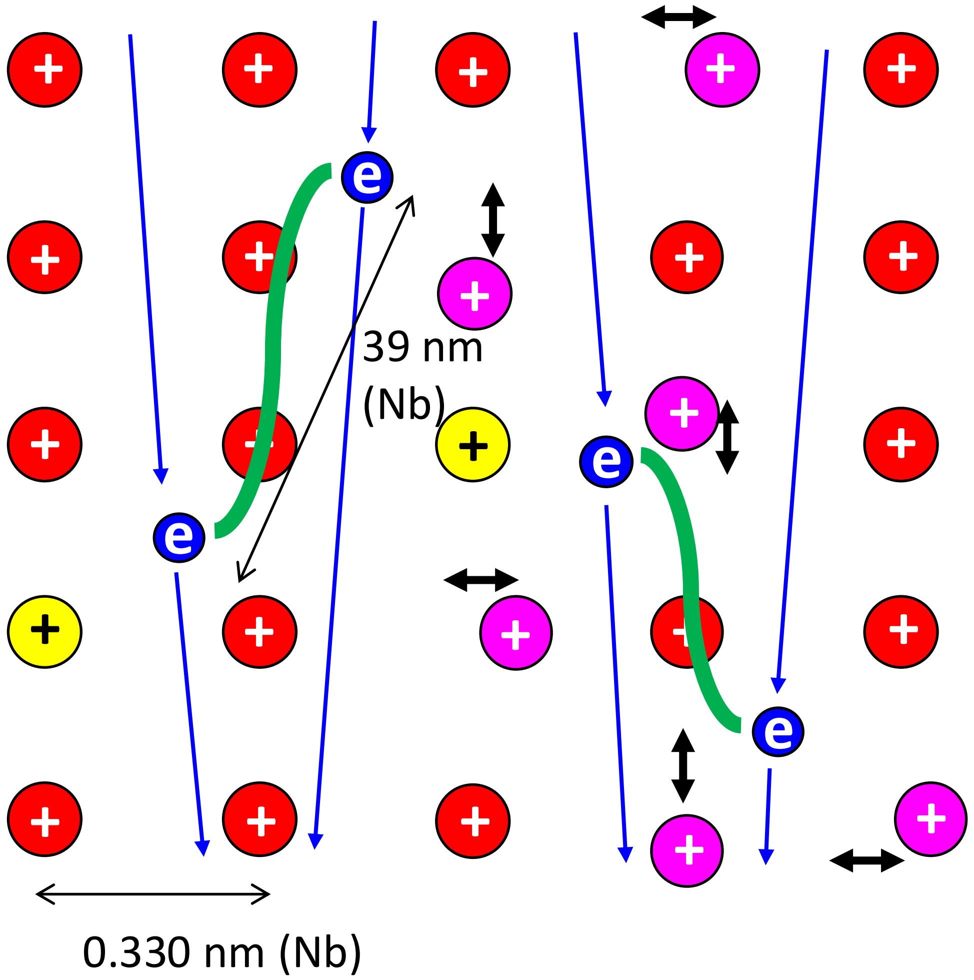
Non-perturbative!

$$\Delta = n(E_F)V \int_{\Delta}^{\hbar\omega_D} \frac{\Delta}{\sqrt{\xi^2 + \Delta^2}} \tanh\left(\frac{1}{2}\frac{\sqrt{\xi^2 + \Delta^2}}{k_B T}\right) d\xi$$

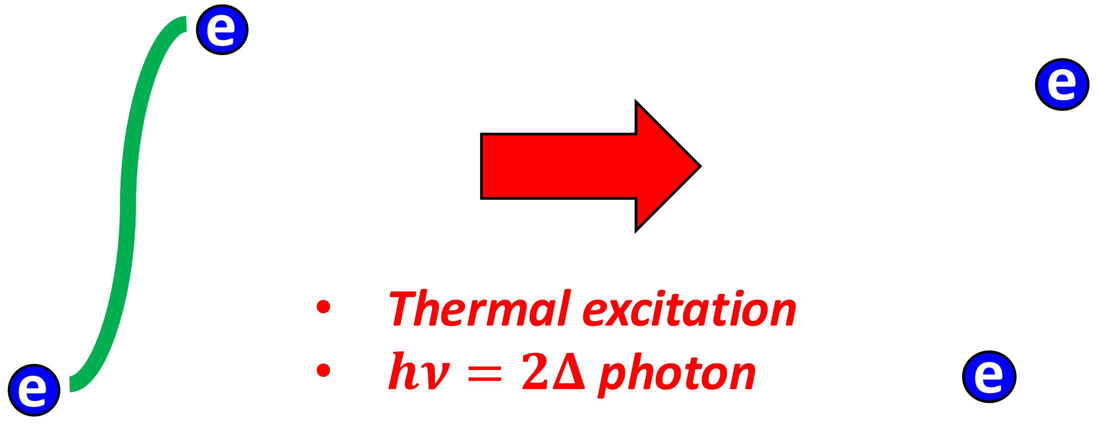
Classical superconductors' attractive potential is from ***longitudinal mode of lattice vibration***



If energy transfer $|\epsilon_{k+q} - \epsilon_k|$ is smaller than phonon energy the interaction is attractive (Flöhlich)
→ Eliashberg's strong coupling superconductor (1960)

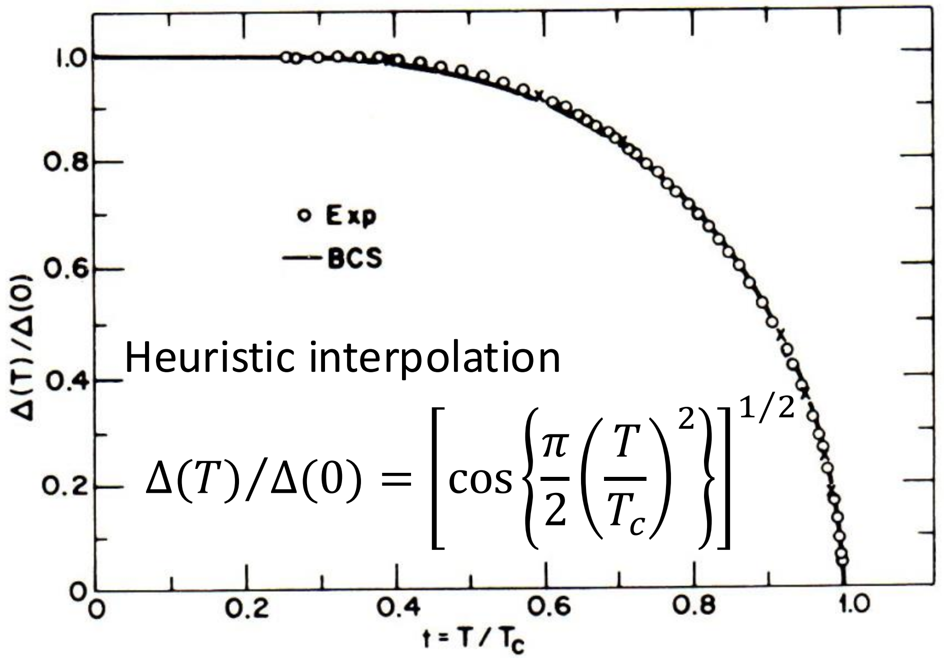


Superconducting gap

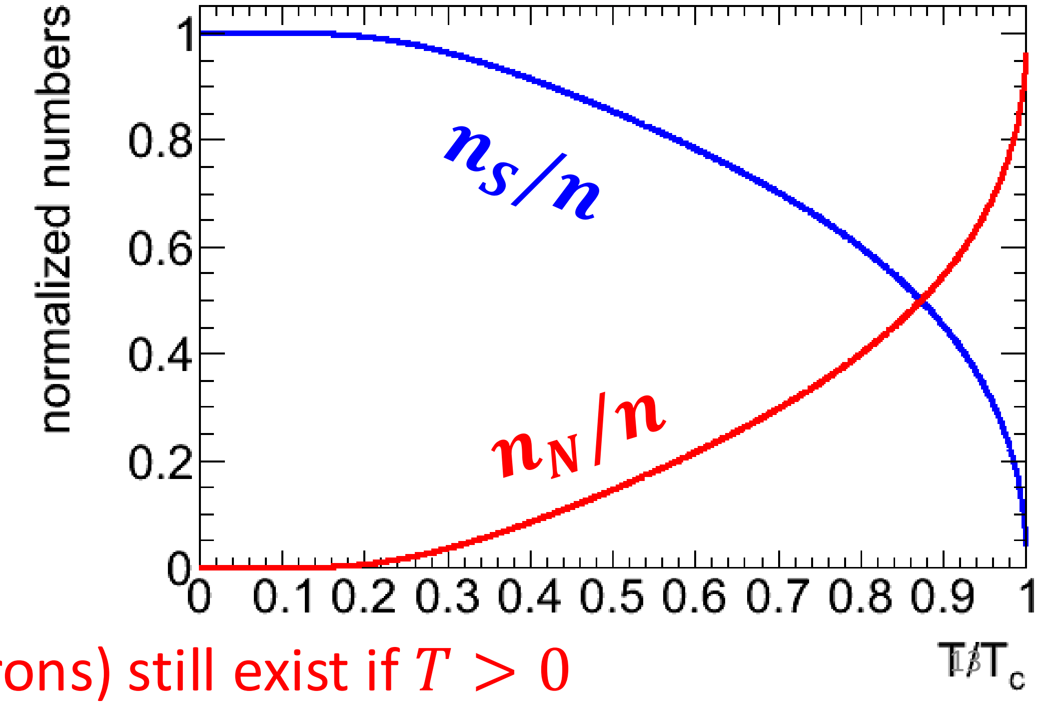


- *Thermal excitation*
- $h\nu = 2\Delta$ *photon*

A Cooper pair ***Two quasi-particles***



$$\Delta(T)/\Delta(0) = \left[\cos\left\{\frac{\pi}{2}\left(\frac{T}{T_c}\right)^2\right\}\right]^{1/2}$$



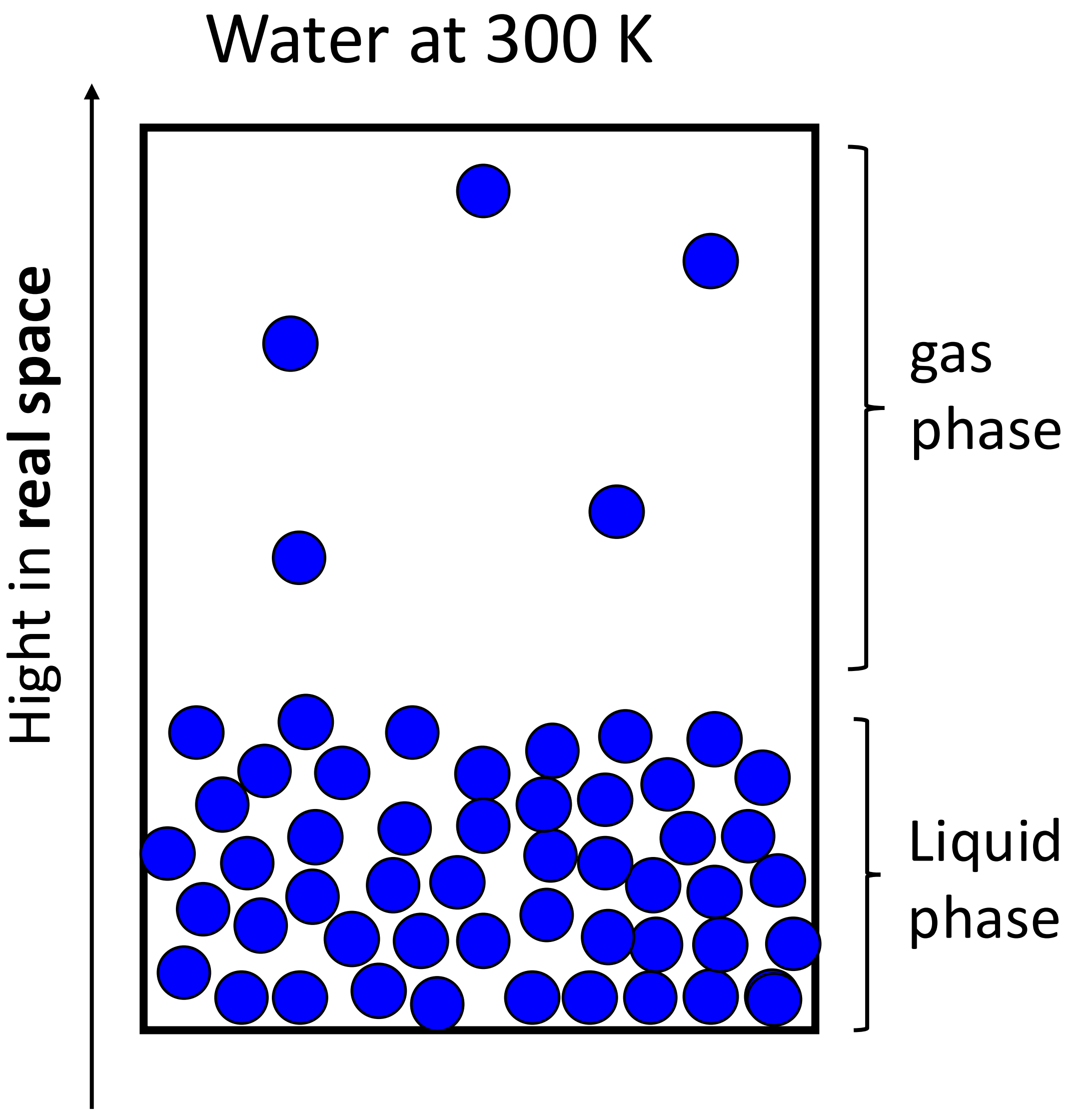
At finite temperature $0 < T < T_c$, these two states are ***in thermal equilibrium***

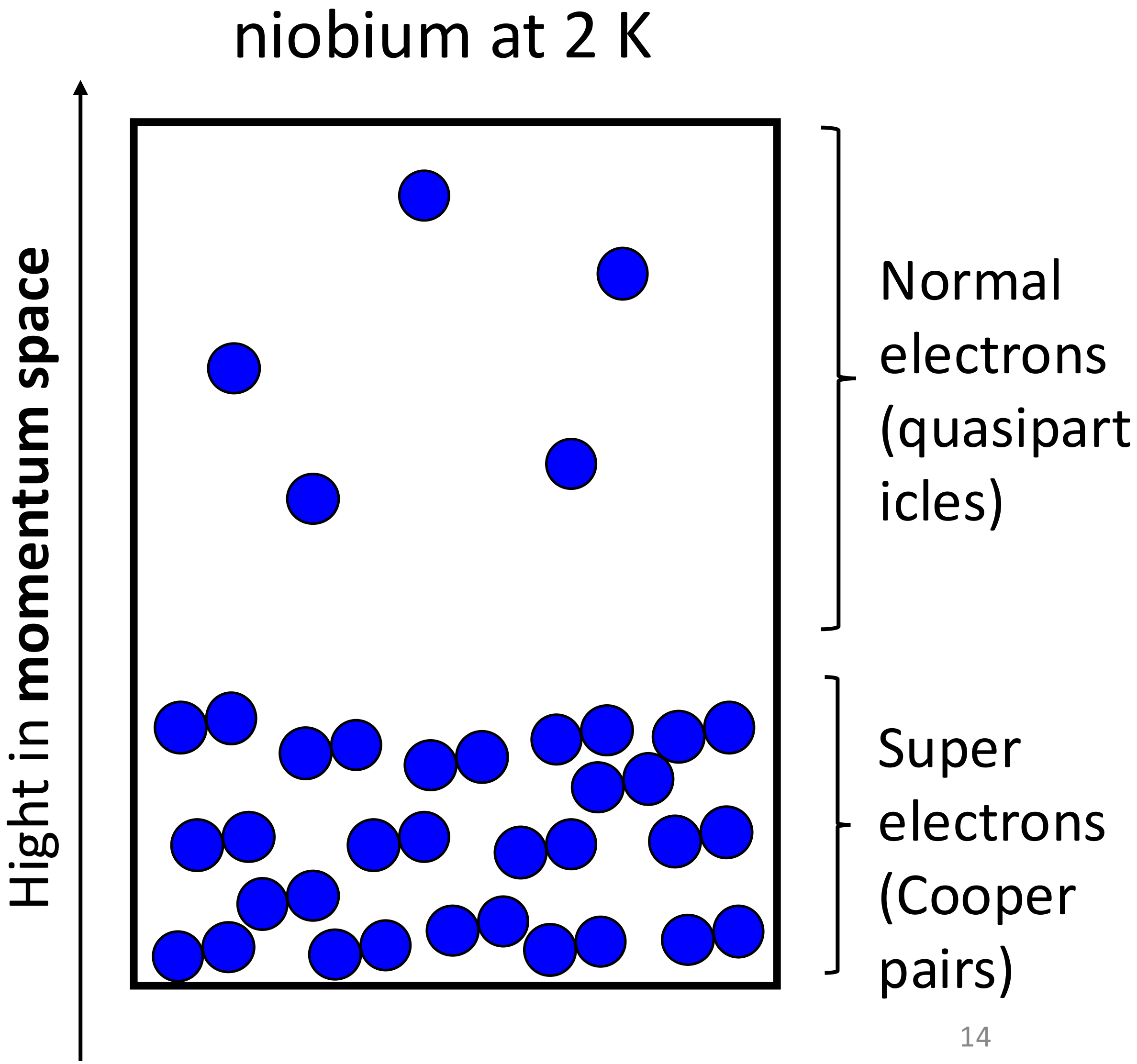
\# of quasiparticles: $n_N \sim \exp\left(-\frac{\Delta}{k_B T}\right)$

\# of electrons in Cooper pairs: $n_S \sim n - n_N$

Quasi-particles (~normal conducting electrons) still exist if $T > 0$

Why normal and super electrons at a time?





Classical two fluid model

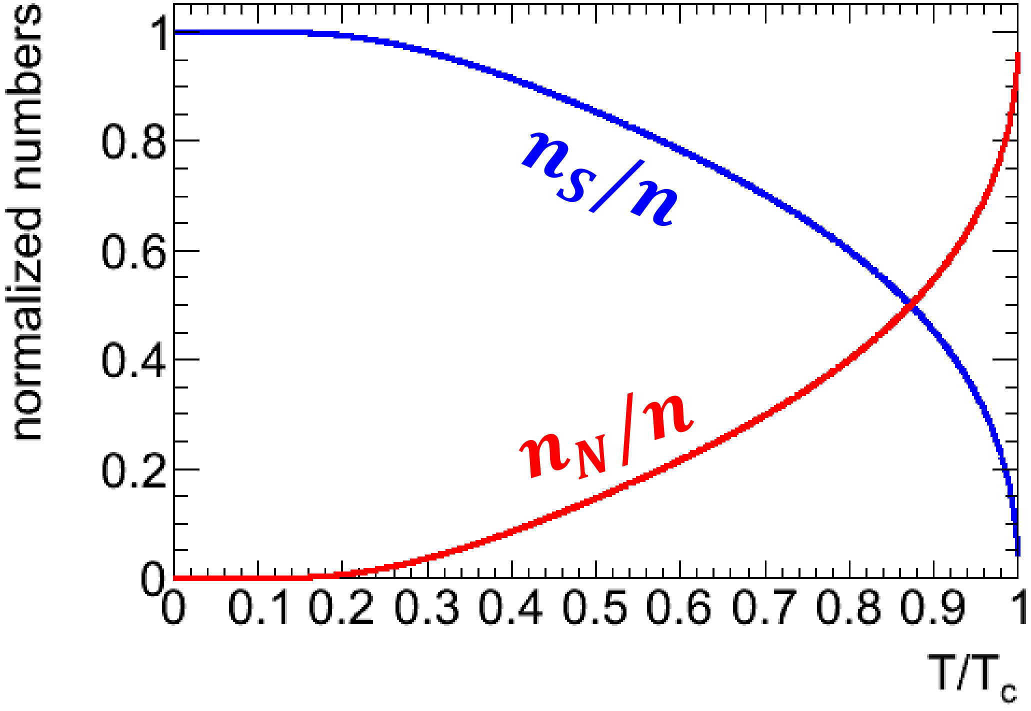
Supercurrent

$$\frac{\partial j_s}{\partial t} - \frac{n_S e^2}{m^*} E = 0$$

$$j_s = j_0 \exp(i\omega t)$$

$$j_s = -i \frac{n_S e^2}{m^* \omega} E$$

$$\equiv \sigma_S$$



Normal current

Ohm's law →

$$j_N = \frac{n_N e^2 \tau}{m^*} E$$

$$\equiv \sigma_N$$

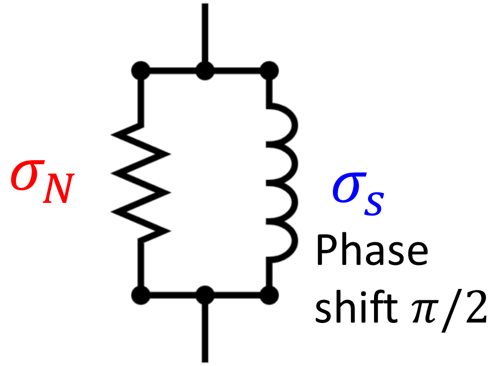
Total current induced by RF

$$j = j_S + j_N \quad \rightarrow \quad j = (\sigma_N - i\sigma_S)E$$

Dissipation by quasi-particles →resistive

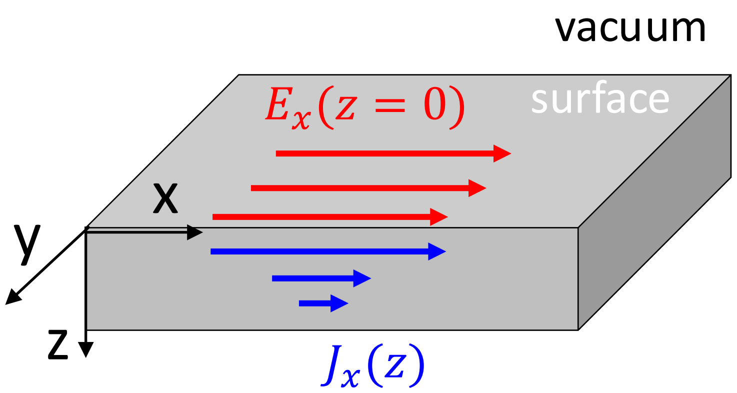
Inertia of Cooper pairs → inductive

Equivalent circuit



Two fluid model

Surface resistance of superconductor



$$\begin{cases} j_x = (\sigma_N - i\sigma_S)E_x + i\omega\varepsilon E_x \\ E_x(z) = E_0 \exp(-z/\lambda_L) \end{cases}$$

$\sigma_N, \omega\epsilon \ll \sigma_S$

Large imaginary part: Perfect diamagnetism in superconductors

$$\rightarrow R_s \equiv \mathrm{Re}\left(\frac{E_x(z=0)}{\int_0^\infty J_x(z)dz}\right) \sim \frac{1}{2}\frac{\sigma_N}{\sigma_S}\sqrt{\frac{\omega\mu_0}{\sigma_S}} = \frac{\mu_0^2}{2}\lambda_L^3\sigma_N\omega^2 > 0$$

$$\sigma_N = \frac{e^2 n_N \tau}{m^*} \propto n_N \propto \exp\left(-\frac{\Delta}{k_B \boldsymbol{T}}\right)$$

<u>Lessons</u>

- Superconducting surface can be modelled by imaginary current
- One origin of the finite R_s of superconductors is quasi-particles
- Quasi-particles are thermally activated from Cooper pairs at $0 < T < T_c$
- R_s exponentially decreases by lower T because quasi-particles are frozen out
- Higher RF frequency increases $R_s \sim \omega^2$ (cf. NC surface resistance is $\propto \omega^{1/2}$)

Penetration depth vs skin depth: similar but totally different origin

Superconductor

Quantum mechanics $\lambda_L = \sqrt{\dfrac{m^*}{n_s e^2 \mu_0}}$

From London equation
(broken gauge symmetry)

$$\nabla^2 \boldsymbol{B} - \frac{1}{\lambda_L^2}\boldsymbol{B} = 0$$

Both **static** magnetic field and **RF** electromagnetic field and currents

For niobium (<9.25K)
$\lambda_L \sim 36$ nm

$\ll$

Normal conductor

$\delta_s = \sqrt{\dfrac{1}{\pi f \mu_0 \sigma}}$ ***From classical electrodynamics***

From a RF screening effect of quasi-particles

$$\boldsymbol{j}_n = \sigma \boldsymbol{E}$$
$$\nabla \times \nabla \times \boldsymbol{E} = -\frac{\partial(\nabla \times B)}{\partial t} \sim \mu_0 \frac{\partial \boldsymbol{j}_n}{\partial t}$$
$$(= -\nabla^2 \boldsymbol{E})$$
$$E = E_0 exp(i2\pi f t)$$

$$\nabla^2 \boldsymbol{E} - \frac{1}{\delta_s^2}\boldsymbol{E} = 0$$

Math looks similar…

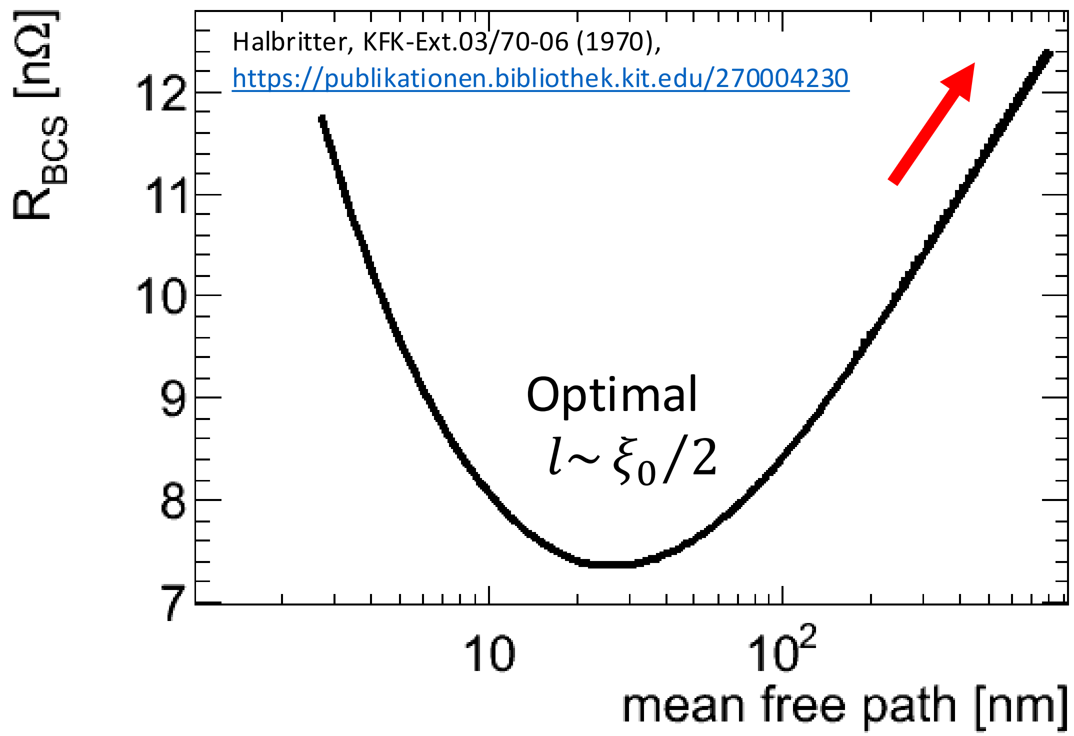
RF electromagnetic fields and currents

For 300K copper and $f = 0.1 - 1$ GHz
$\delta > 2$ µm

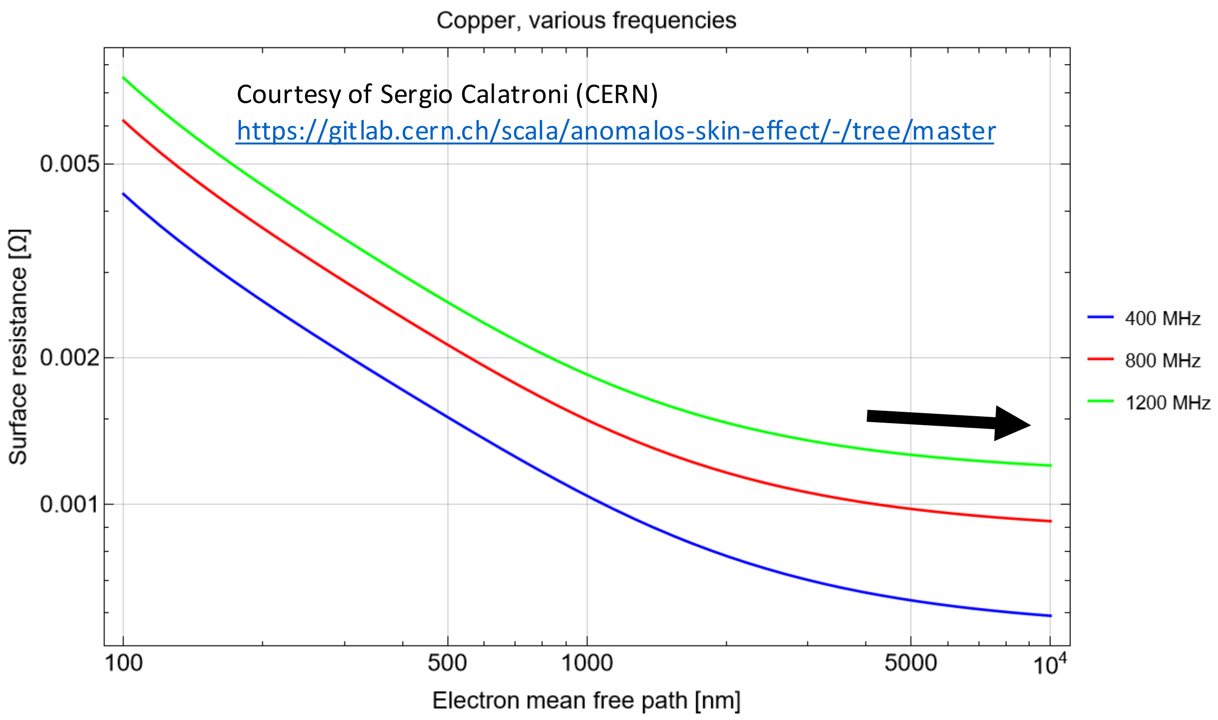
Anomalous skin effect

Clean material (long l, high RRR) does not improve surface resistance at high frequency microwaves in cryogenics ($l > \delta, \xi_0$ → nonlocality effect)

Superconducting niobium



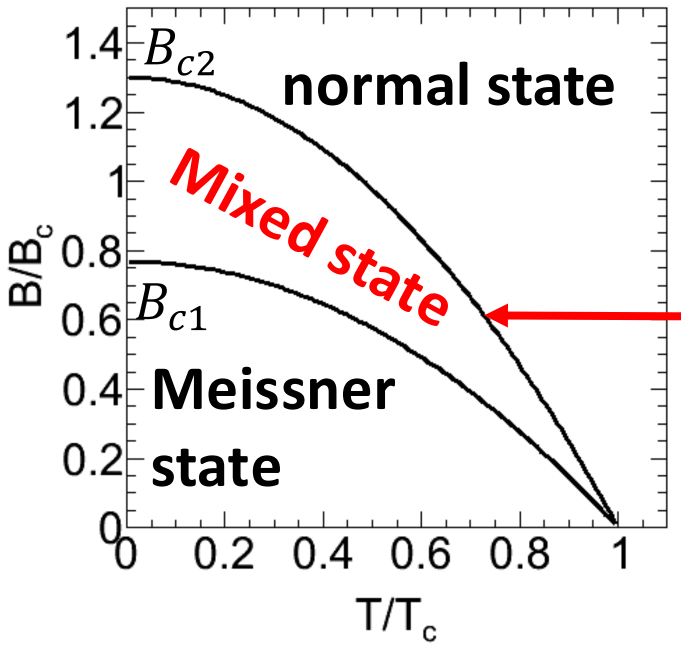
Normal conducting copper



Dirty (→ short mean free path within λ or δ) is better (for superconductors) or enough (normal conductors)

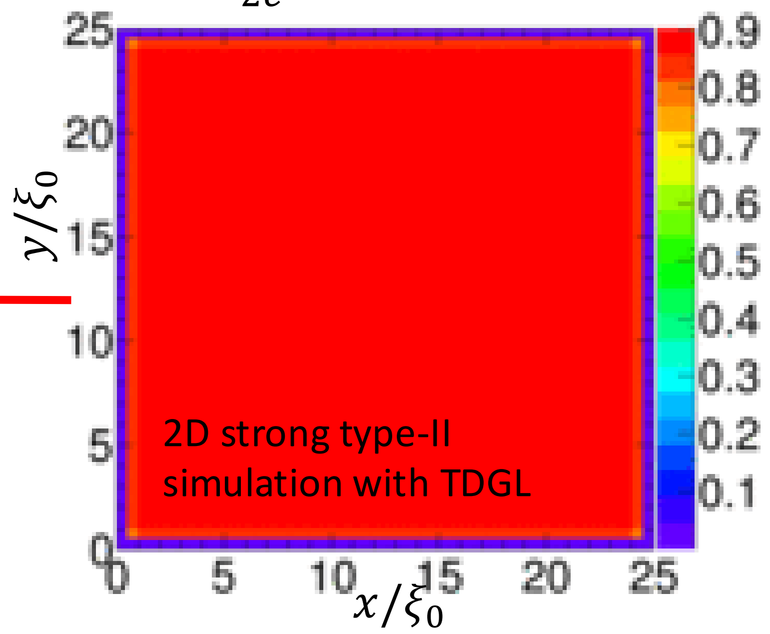
R_{res} contribution from magnetic flux oscillation

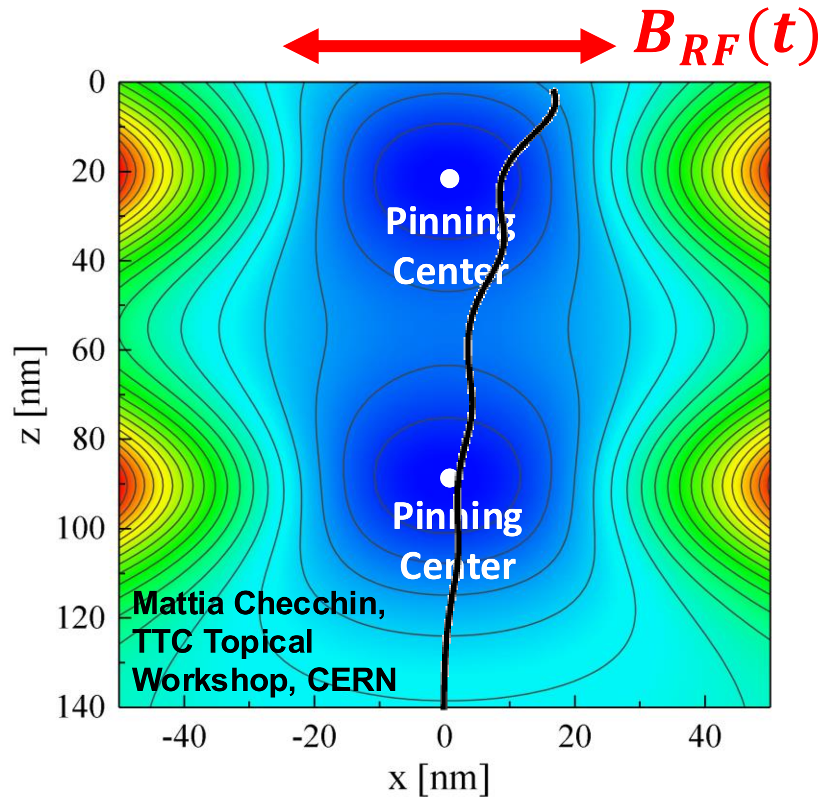
$$\kappa = \frac{\lambda}{\xi} > \frac{1}{\sqrt{2}} = 0.71$$



Quantized DC flux

$$\Phi_0 = \frac{h}{2e} = 2.07 \times 10^{-15}\ \mathrm{Wb}$$





DC (external B-field) + RF (axion-induced)

$$M\frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \eta\frac{\partial \boldsymbol{u}}{\partial t} - \epsilon\frac{\partial^2 \boldsymbol{u}}{\partial z^2} + \nabla U(z,u) = \boldsymbol{J}_{RF}(z,u) \times \boldsymbol{B}_{\mathrm{ext}}$$

Effective inertia | Effective viscosity | Effective tension | **Pinning potential** | **Lorentz force drives flux oscillation**

This flux oscillation can cause substantial reduction in Q

Superconductor is ***protected*** against ***parallel*** magnetic fields

<u>Solving London equation with the image force term</u>

$$\nabla^2 H(x,z) - \frac{1}{\lambda^2} H(x,z) = -\frac{\phi_0}{\mu_0 \lambda^2}[\delta(x)\delta(z-z_0) - \delta(x)\delta(z+z_0)]$$

Results in two terms

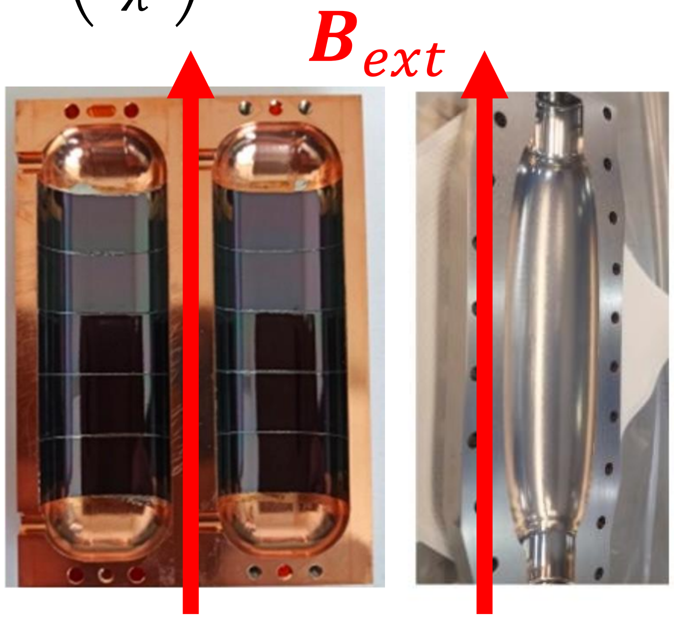
1. External field term which attracts the parallel flux

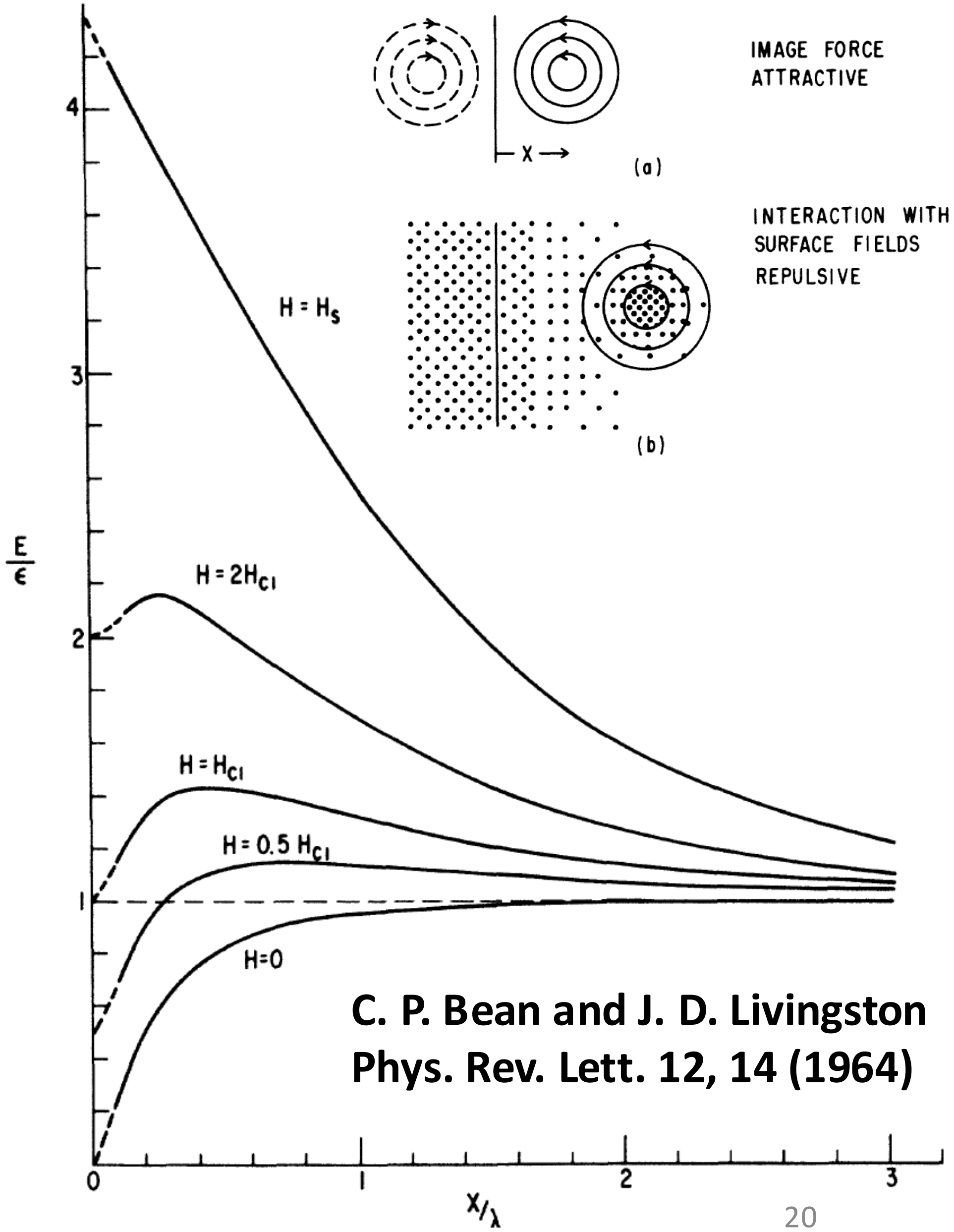
$$f_1 = \frac{\phi_0 H_0}{\lambda} \exp\left(-\frac{z_0}{\lambda}\right)$$

2. Image force term which **expels the parallel flux**

$$f_2(x) = \frac{\phi_0}{2\pi\mu_0\lambda^3} K_1\left(\frac{2z_0}{\lambda}\right)$$

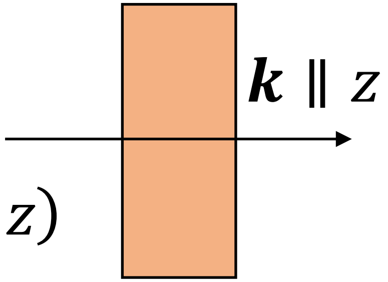
- SC materials is relatively robust against parallel B-field thanks to the BL surface barrier
- The SC cavities are optimized to avoid SC areas perpendicular to the B-field





C. P. Bean and J. D. Livingston
Phys. Rev. Lett. 12, 14 (1964)

Propagation of microwaves in dielectrics



$$\nabla \times \boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad \boldsymbol{D} = \varepsilon \boldsymbol{E} \qquad \boldsymbol{j} = \sigma \boldsymbol{E}$$

$$\varepsilon = \varepsilon' - i\varepsilon'' \qquad \boldsymbol{E} = \boldsymbol{E}_0 \exp(i\omega t - i((k' - k'')z))$$

dielectric loss

$$\rightarrow \nabla \times \boldsymbol{H} = \sigma \boldsymbol{E} + (\varepsilon' - i\varepsilon'') \times i\omega \boldsymbol{E} = (\omega\varepsilon'' + \sigma)\boldsymbol{E} + i\omega\epsilon'\boldsymbol{E}$$

Ohmic loss

$$= i\omega\varepsilon'[1 - i\tan\delta_e]\,\boldsymbol{E}$$

$$\tan\delta_e \equiv \frac{\omega\varepsilon'' + \sigma}{\omega\varepsilon'}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \boldsymbol{B} = \mu \boldsymbol{H} \qquad \mu = \mu' - i\mu'' = \mu'[1 - i\tan\delta_m]$$

magnetic loss

$$\tan\delta_m \equiv \frac{\mu''}{\mu'} > 0$$

For ferrimagnetic material (ferrite)

Attenuation $I(z) = I_0 \exp\left[-\frac{z}{z_0}\right]$

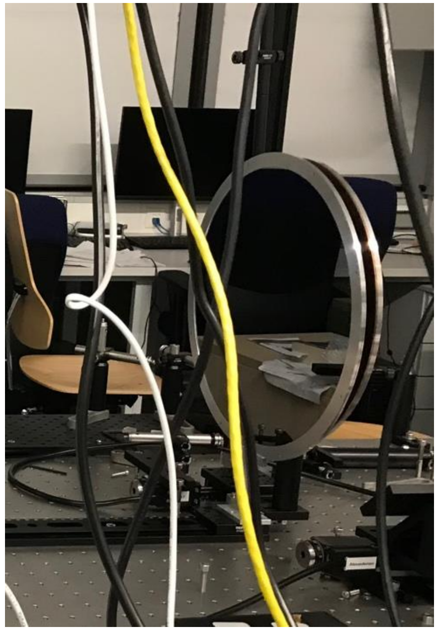
for pure dielectric $\mu'' = 0, \sigma = 0$

$$z_0 \sim \frac{1}{2\omega}\sqrt{\frac{1}{\varepsilon'\mu_0}}\sqrt{\frac{2}{-1 + \sqrt{1 + \tan^2\delta_e}}}$$

Semi-conductor vs insulator

$$\tan\delta_e = \tan\delta_d + \frac{1}{\rho\omega\epsilon'} \xrightarrow{\rho\to\infty} \tan\delta_d$$

High-resistivity Si crystal



J. Krupkam P. Kaminski, L. Jensen, "high Q-factor dielectric resonators made of semi-insulating silicon for millimter wave applications"

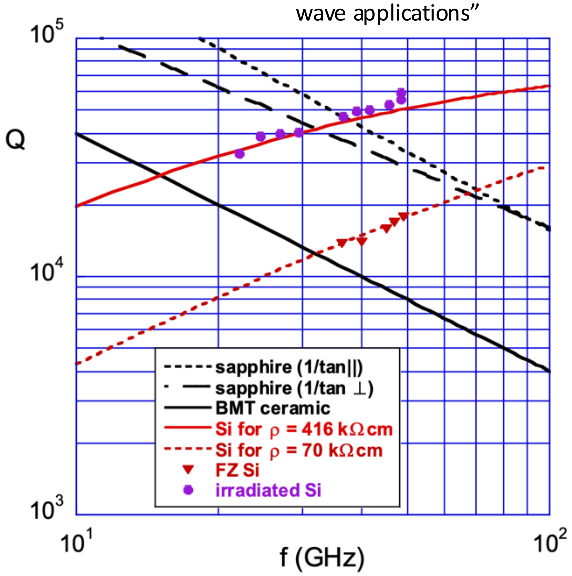


Fig.3. Q-factors (1/tanδ) as a function of frequency for dielectric resonators made of various low dielectric loss materials at room temperatures

Sapphire crystal

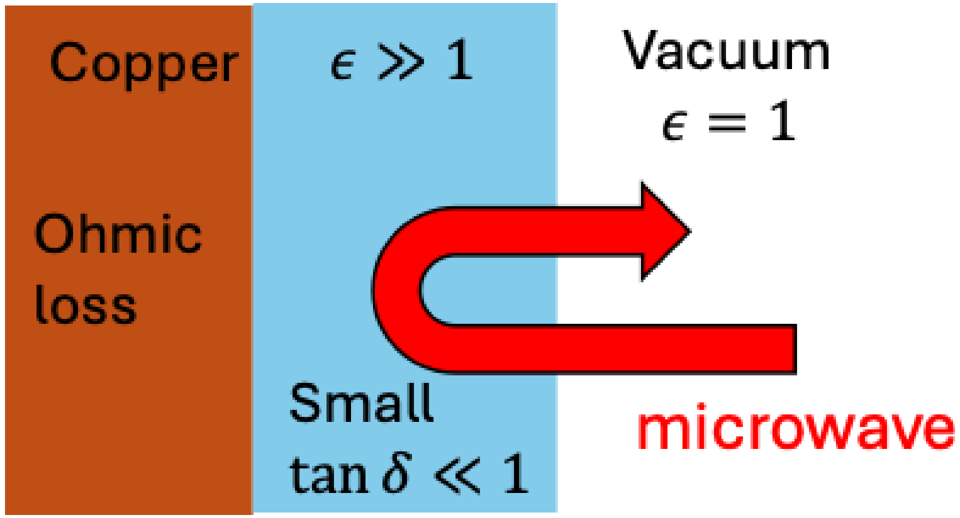
SurfaceNet

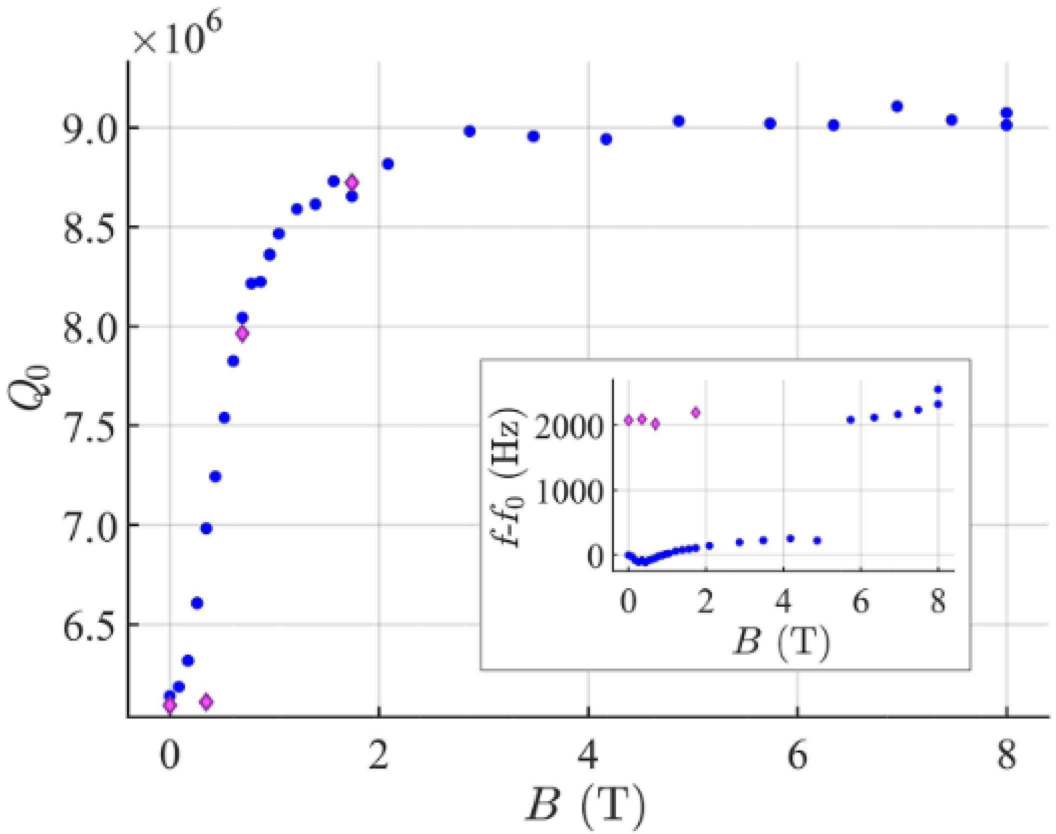
- Sapphire has $\tan\delta \sim 10^{-5}$ at 300K and $\tan\delta \sim 10^{-6}$ in cryogenic environment
- Normal Si is loss at 300K → carriers frozen in cryogenics → $\tan\delta \sim 10^{-5}$ in cryogenics
- High-resistivity Si can have excellent loss even at 300 K
- Mechanical properties: sapphire is harder, heavier, and more bulky

Interesting observation in sapphire resonators

$$Q_0 \sim \frac{1}{\tan\delta}$$

Courtesy: Caterina Braggio, "Status of the QUAX experiment"





- Quality factor of sapphire resonator was improved under static magnetic field in cryogenic environment
- The loss mechanism seems supressed by the B-field unlike SC cavities

Part 2: classical detection scheme

- Boundary conditions
  - Normal conductors
  - Superconductors
  - Dielectric materials: insulator and semi-conductors
- Microwave resonators
  - Waveguide and transmission line
  - Resonant cavity
  - Fabry-Pérot resonator
- Analog and digital system
  - Amplifier, circulator, mixer & analogue down-conversion
  - I/Q sampling & digital down-conversion
- Data processing and noise
  - FFT: coherent and incoherent integral
  - Narrowband and broadband
  - Thermal noise and standard quantum limit
- Conclusion of part 2

Propagation in a free space vs waveguides

Free space $(E_z, B_z) \neq (0,0)$	Waveguide $(E_z, B_z) \neq (0,0)$	Transmission $(E_z, B_z) = (0,0)$

3D Helmholtz equation

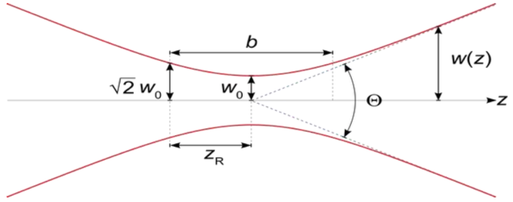
$$(\nabla^2 + k^2)\psi(x,y,z)e^{-ikz} = 0$$

With $\psi \to 0$ $(r \to \infty)$

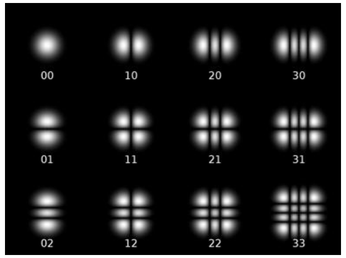
Paraxial approximation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - 2ik\frac{\partial\psi}{\partial z} = 0$$

→ **Gaussian beam(s)**



Conventionally called TEM but $(E_z, B_z) \neq (0,0)$



2D Helmholtz equation

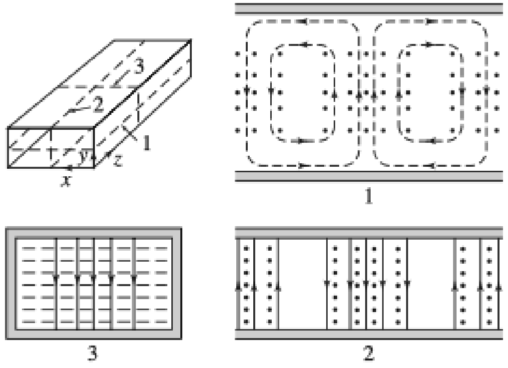
$$(\nabla_t^2 + k_c^2)E_z(x,y) = 0$$

With $E_z\big|_S = 0$ TM waves

or

$$(\nabla_t^2 + k_c^2)B_z(x,y) = 0$$

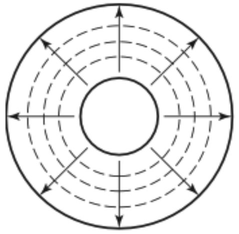
With $\left.\frac{\partial B_z}{\partial n}\right|_S = 0$ TE waves



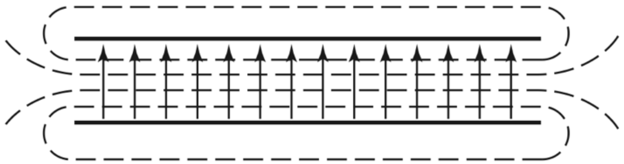
2D Laplace equation

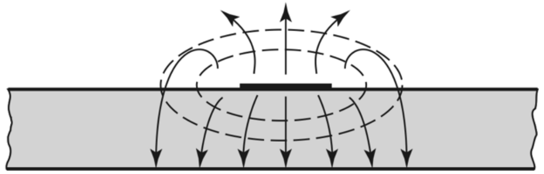
$$\nabla_t^2\,\Phi(x,y) = 0$$

With $\Phi(x,y) = ?$ on boundary

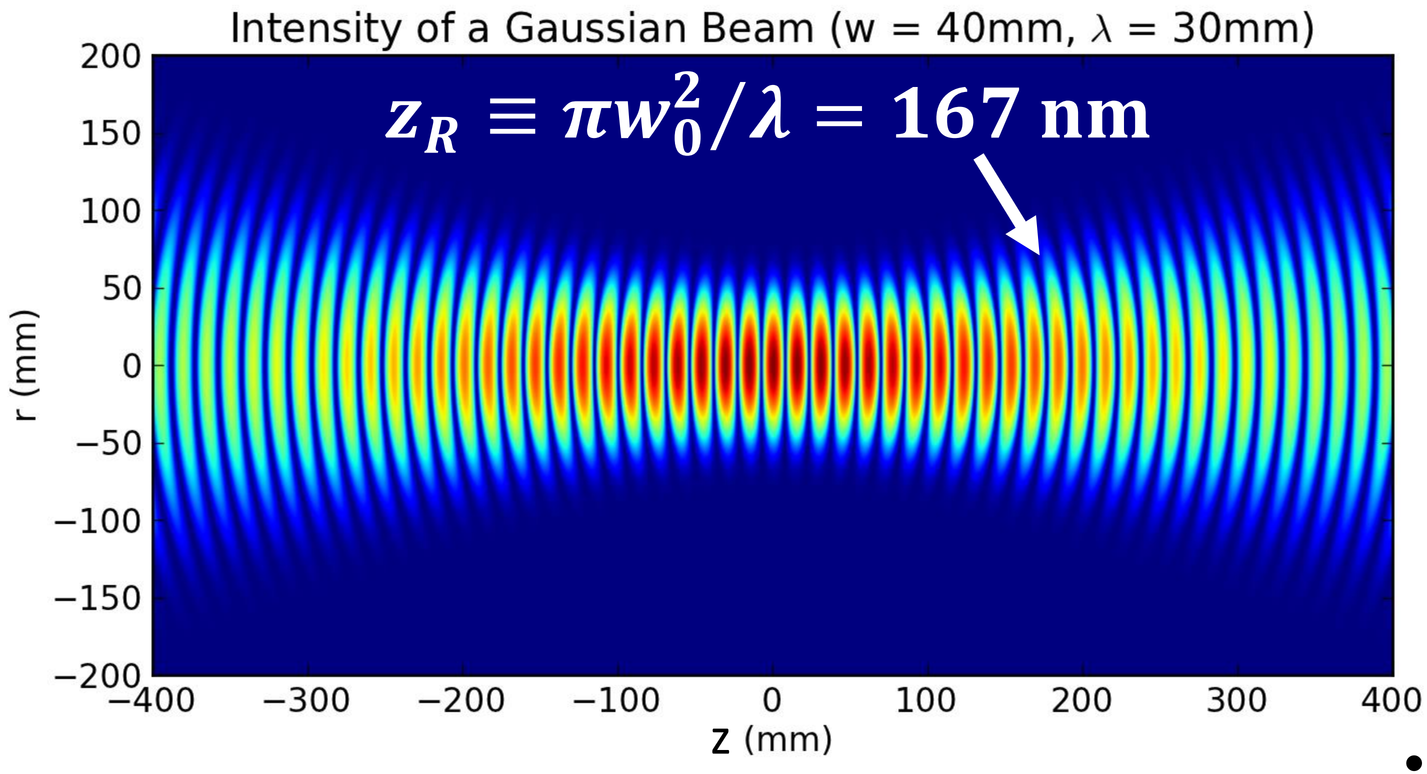


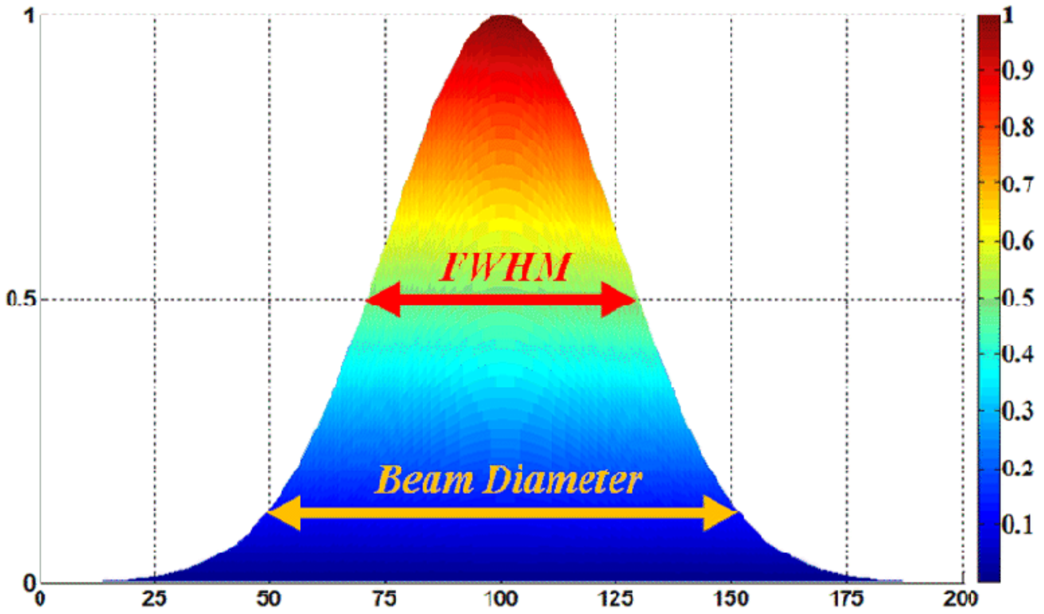
TEM waves

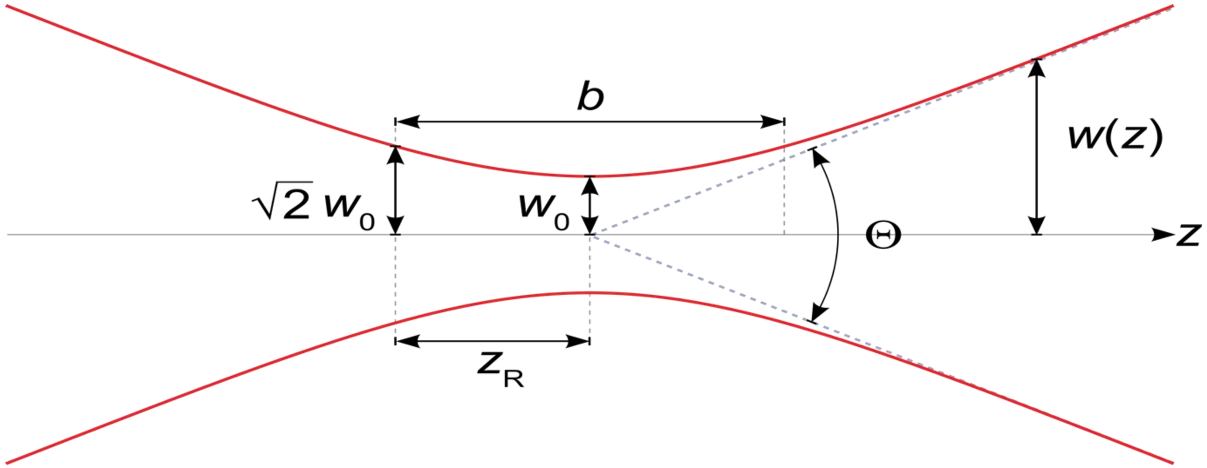




Visualization of Gaussian beam from wiki

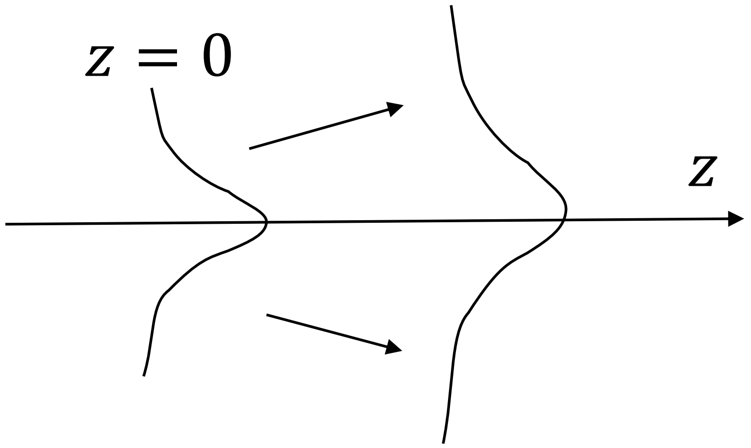






- Beam waist's size and position with wavelength unique determines the Gaussian beam!
- $w(z)$ is 2σ in ***power not amplitude***
- A Gaussian beam is just **a** solution
  → infinite other solutions exist!

Diffraction conserves Gaussian beam



$$E(x,y,z) = \frac{ie^{-ikz}}{\lambda z}\iint_{-\infty}^{\infty} E(x',y',0)\exp\left[\frac{-ik\{(x-x')^2+(y-y')^2\}}{2z}\right]dx'\,dy'$$

At $z = 0$ $\quad E(x,y,0) = E_0\exp(-(x^2+y^2)/w_0^2)$

$$\rightarrow E(x,y,z) = \frac{E_0 e^{-ikz}}{1-\frac{i\lambda z}{\pi w_0^2}}\exp\left(-\frac{ik}{2R(z)}r^2 - \frac{r^2}{w(z)^2}\right)$$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2-\beta x}dx = \sqrt{\frac{\pi}{\alpha}}\exp\left(\frac{\beta^2}{4\alpha}\right)$$

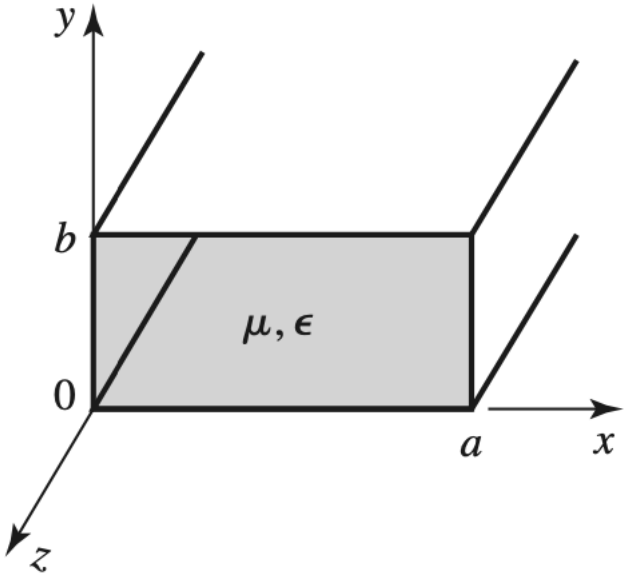
$$R(z) = z + \frac{1}{z}\left(\frac{\pi w_0^2}{\lambda}\right)^2 \qquad w(z)^2 = w_0^2\left(1+\left(\frac{z\lambda}{\pi w_0^2}\right)^2\right)$$

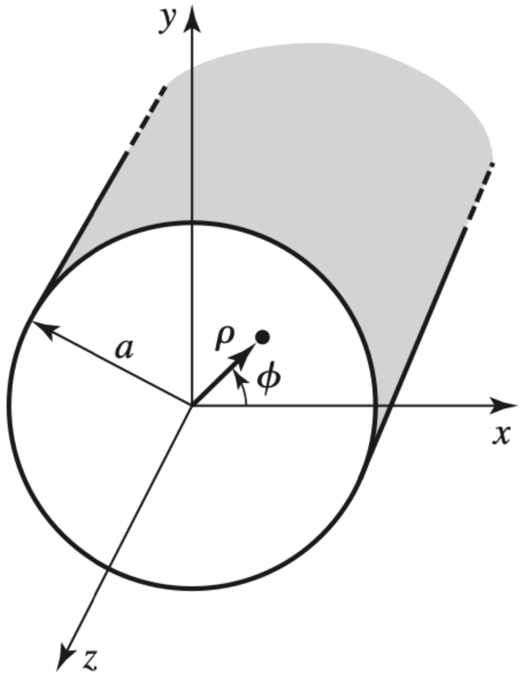
Diffraction = Fourier transform → Fourier transform of Gaussian is Gaussian

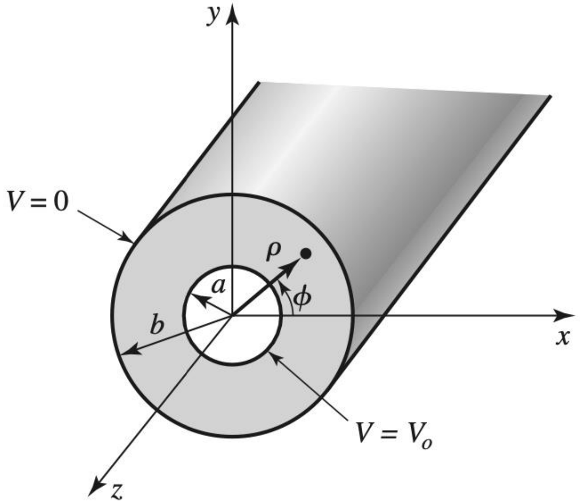
The distribution stays as Gaussian for any z

→ Gaussian beam propagation is an eigenmode of free space Maxwell equation

Three most basic waveguide / transmission lines







TE
TM
modes

TE
TM
modes

TEM
TE
TM
modes

→ Min frequency of propagation (cut-off)

→ Max frequency of propagation

Cutoff frequency

Waves propagate as $\psi \sim e^{\gamma z}$ with the propagation constant $\gamma = \alpha + i\beta \in \mathbb{C}$

The solution of differential equations gave digitized $k_{c,nm} \in \mathbb{R}$ that determines

$$\gamma_{mn}^2 = k_{c,nm}^2 - k_0^2 = k_{c,nm}^2 - \mu\varepsilon\omega^2$$

For a propagating mode in a lossless waveguide, $\gamma_{mn} = i\beta_{mn}$ with $\beta_{mn} \in \mathbb{R}$

$$\beta_{mn} = \sqrt{\mu\varepsilon\omega^2 - k_{c,nm}^2} \rightarrow \mu\varepsilon\omega^2 > k_{c,nm}^2$$

$$\rightarrow f = \frac{\omega}{2\pi} > \frac{k_{c,nm}}{2\pi\sqrt{\mu\varepsilon}} \equiv f_{c,nm}$$

Otherwise ($f < f_{c,nm}$) evanescence mode damps inside waveguide with $\psi \sim e^{-\alpha z}$

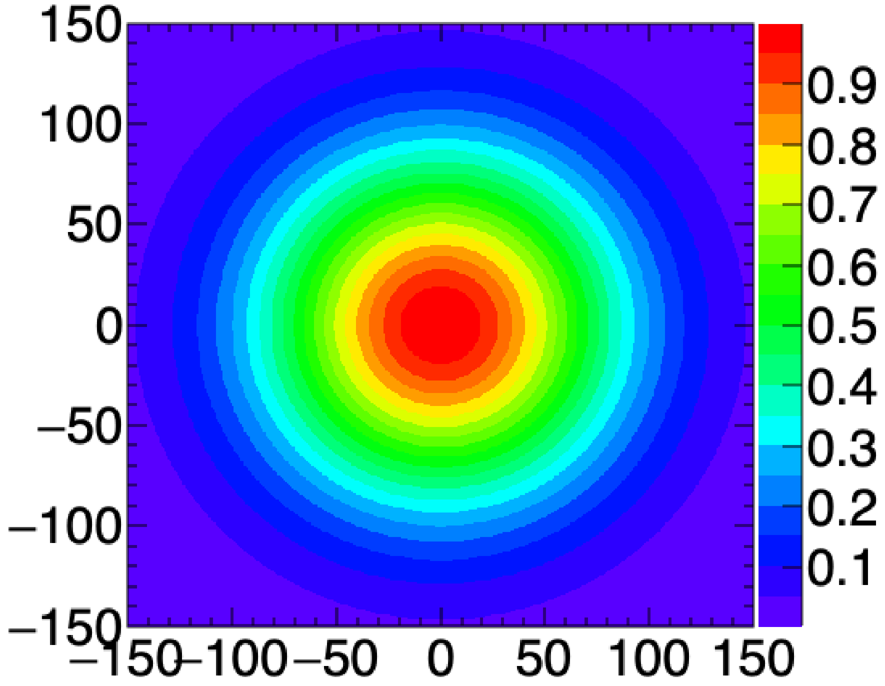
Conventionally, waveguide is selected between two cut-off frequency

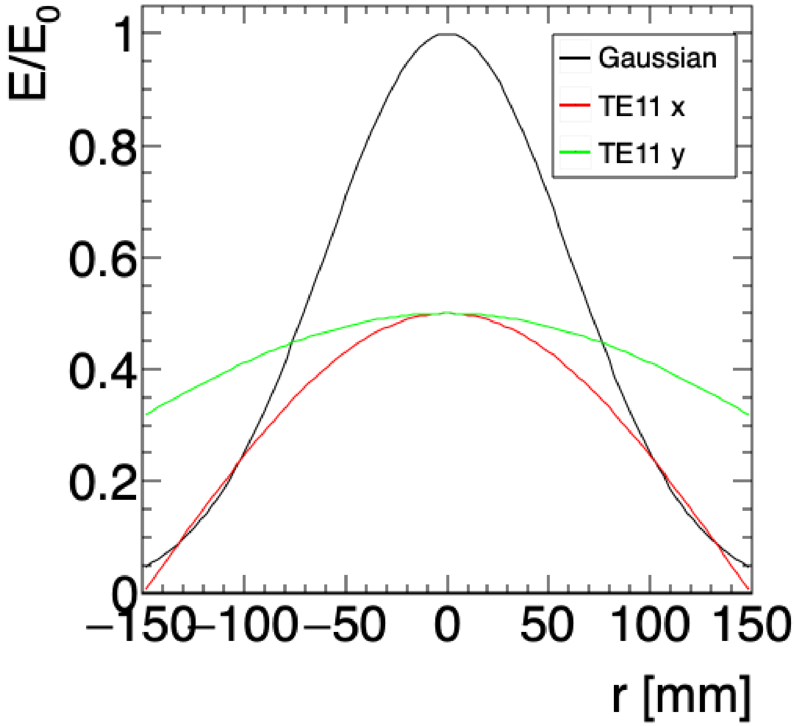
→ Only TE10 (rectangular) or TE11 (cylinder) is above the cut-off for the given frequency f

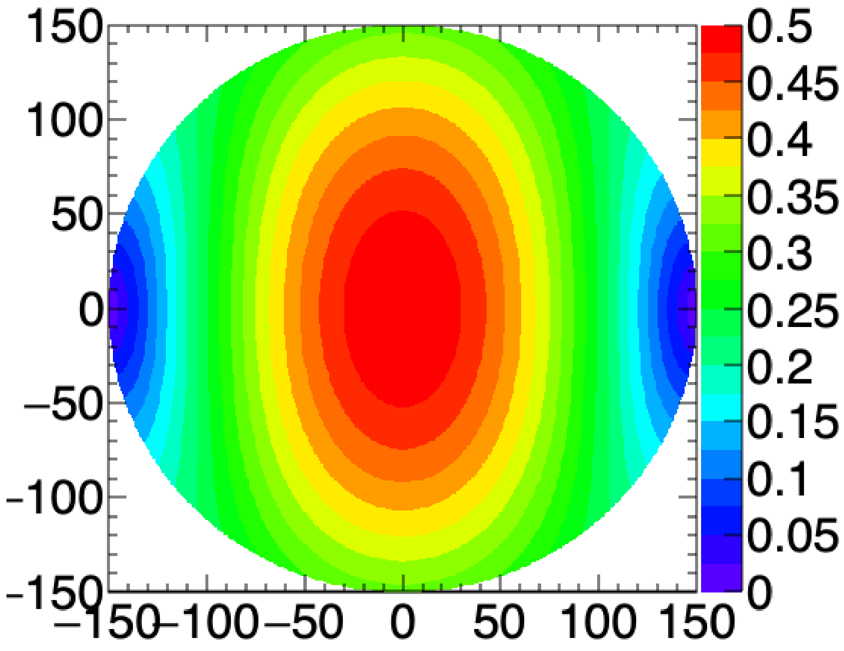
Gaussian beam in free space vs circular TE11

Gaussian beam $w_0 = 85\text{mm}$

TE11 in $a = 150$ mm WG





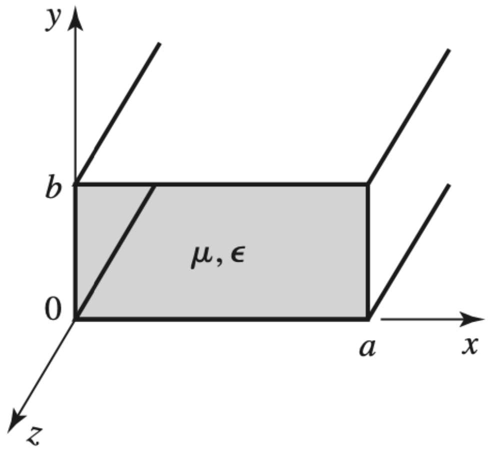


Relatively similar shape → horn antenna from circular TE11 to a Gaussian beam with relatively nice efficiency (less side robes) → HE11 with corrugation is even better

Plane wave expansion and group velocity

$$\begin{cases} E_x = 0 \\ E_y = \frac{i\mu\omega\pi}{k_c^2 a} H_0 \cos\left(\frac{\pi x}{a}\right) e^{i\beta z - i\omega t} \\ E_z = 0 \end{cases} \qquad \begin{cases} H_x = -\frac{i\beta\pi}{k_c^2 a} H_0 \sin\left(\frac{\pi x}{a}\right) e^{i\beta z - i\omega t} \\ H_y = 0 \\ H_z = H_0 \cos\left(\frac{\pi x}{a}\right) e^{i\beta z - i\omega t} \end{cases}$$

$$\beta = \sqrt{(\omega/c)^2 - k_c^2}$$



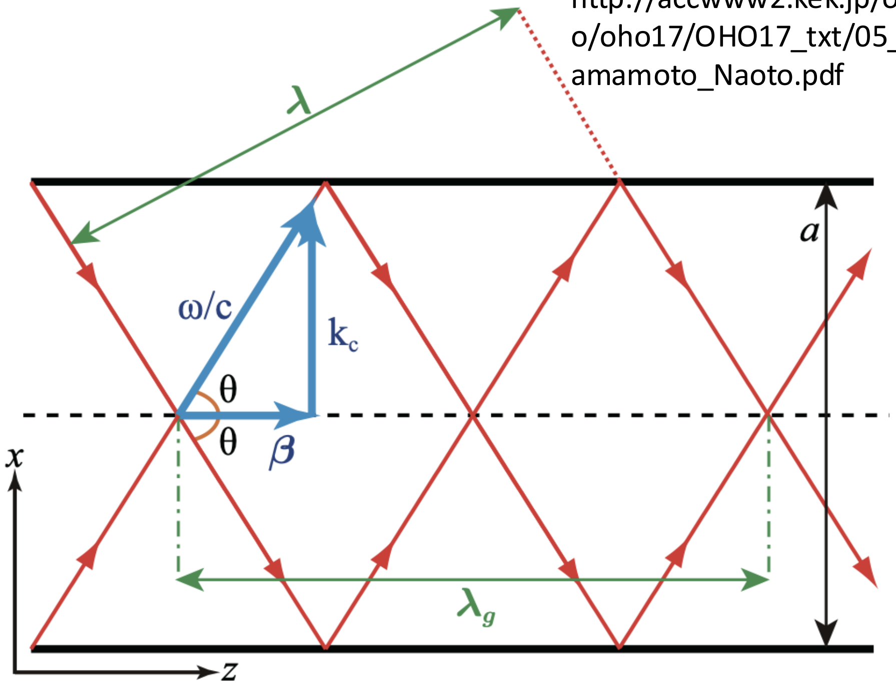
$$\equiv \cos(k_x x) = \frac{1}{2}\left[e^{ik_x x} + e^{-ik_x x}\right]$$

$$H_z = \frac{H_0}{2}\left[e^{i(k_x x + \beta z - \omega t)} + e^{i(-k_x x + \beta z - \omega t)}\right]$$

$$k_x = \frac{\pi}{a} \equiv k_c$$

- Superposition of two propagating planes waves
- Counter propagating in x direction

http://accwww2.kek.jp/oho/oho17/OHO17_txt/05_Yamamoto_Naoto.pdf



The velocity of the plane wave propagated in a tilted angle

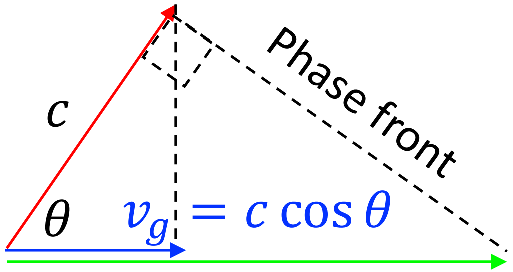
$$\theta = \tan^{-1}(k_c/\beta)$$

is speed of light in the medium $c = \omega/k$ (zig-zag with light speed)

<u>Group velocity: energy flow speed</u>

$$v_g = \frac{P}{E} = c\cos\theta$$

$$v_g v_p = c^2$$

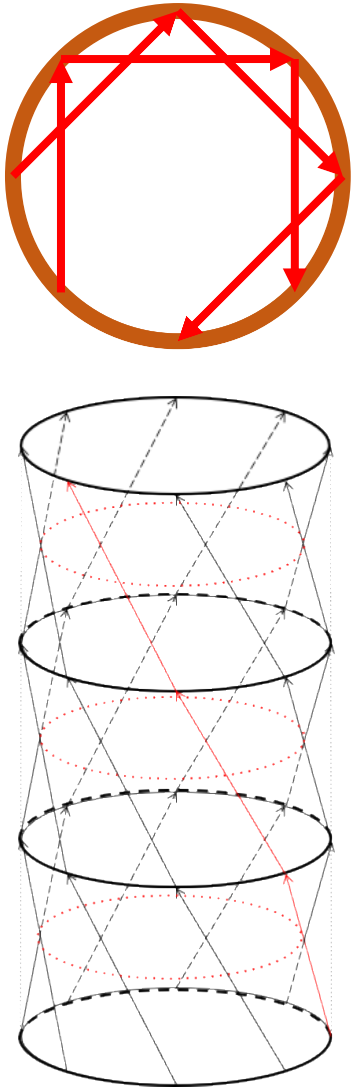


$$v_p = \frac{c}{\cos\theta}$$

Plane wave expansion of cylindrical modes

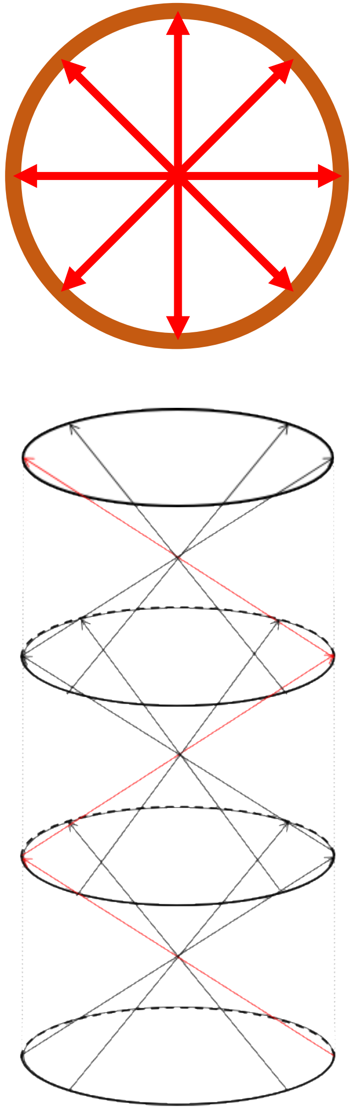
TE_{nm}, TM_{nm} $(n \neq 0)$

Rotating two polarizations

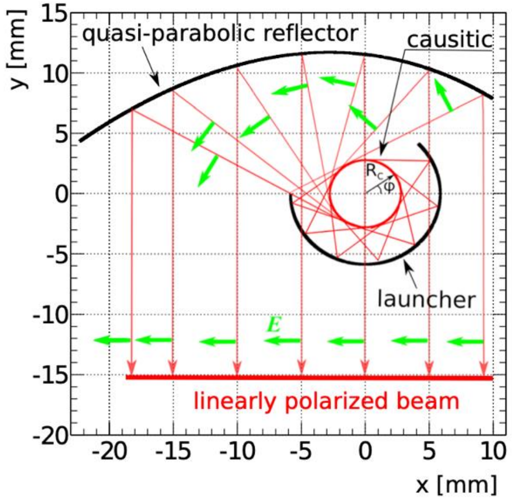


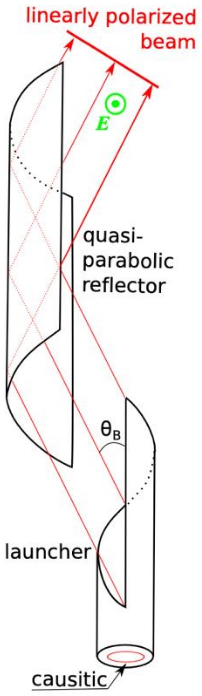
TE_{0m}, TM_{0m}

Shrink/expand Degenerated polarization

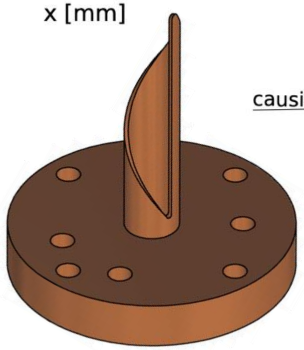


Important if one wants to design waveguide to bi-Gaussian beam transition (horn antenna is an option for TE_{11} only)

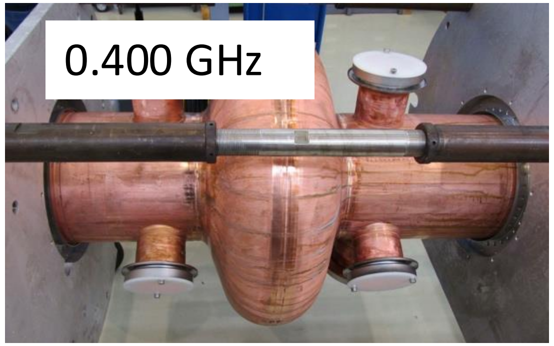


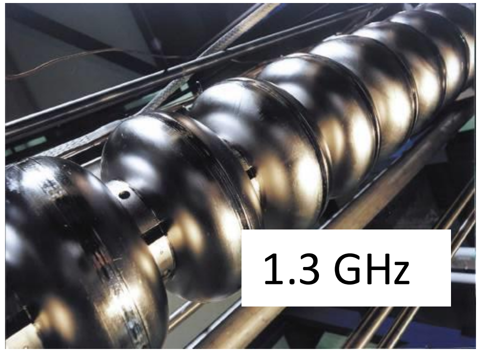


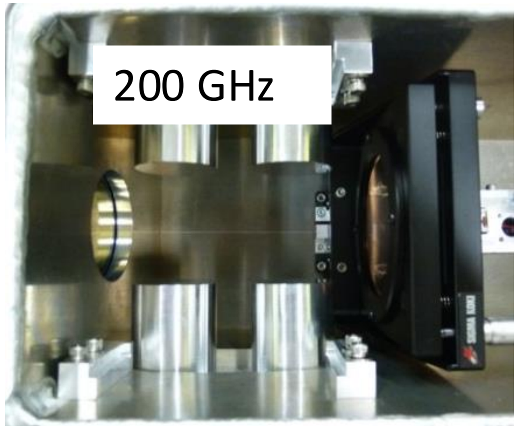
AM, PhD thesis

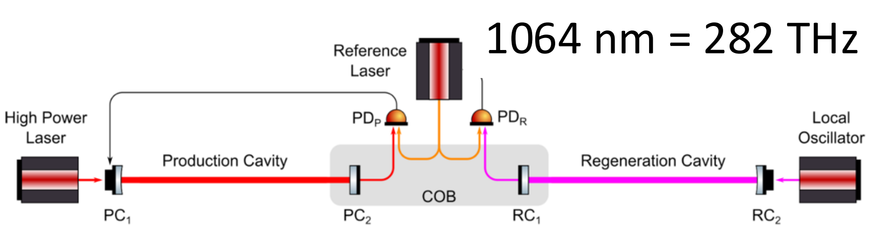


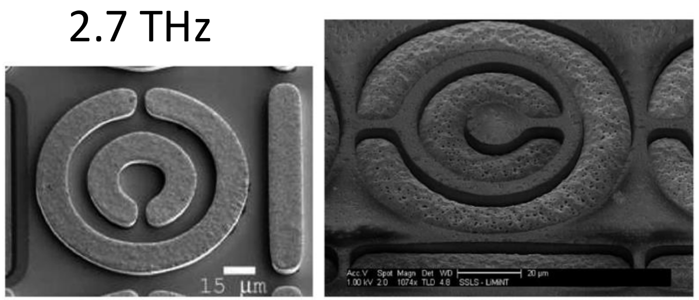
Different types of resonators

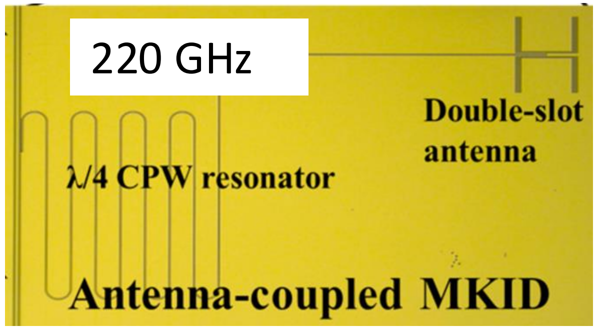


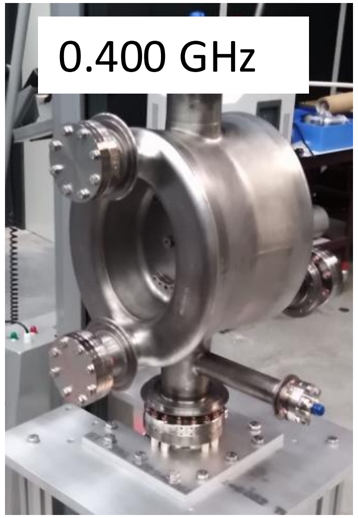


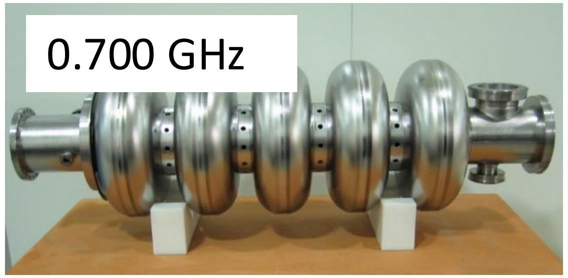


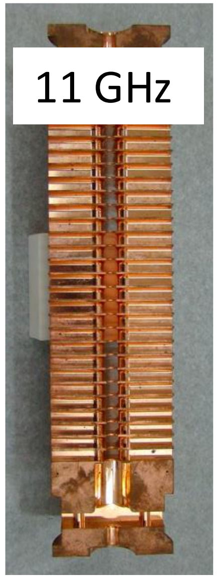


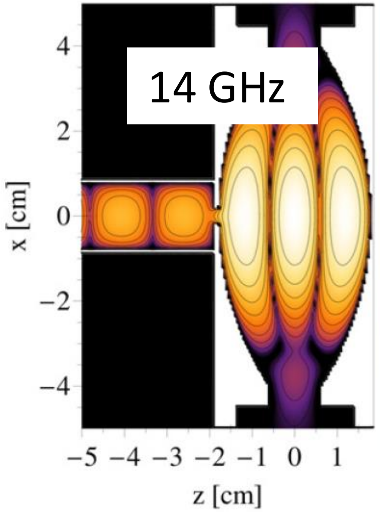


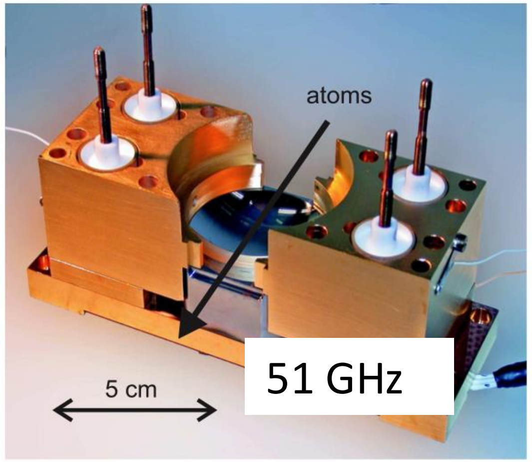


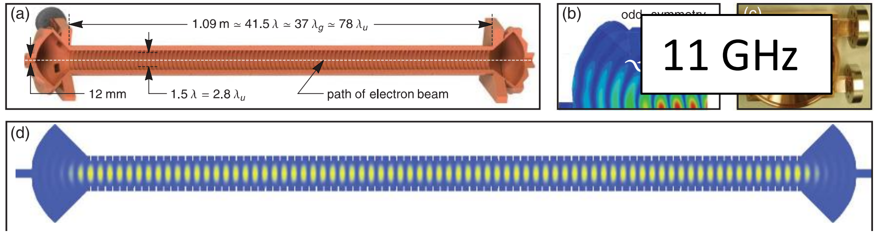




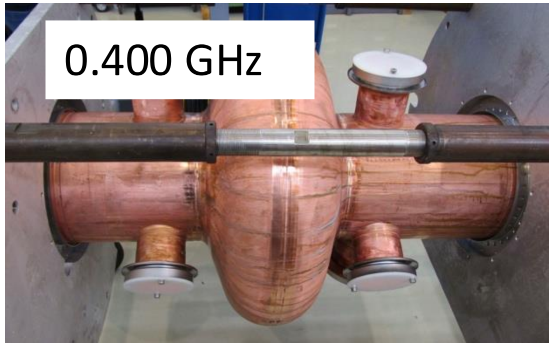


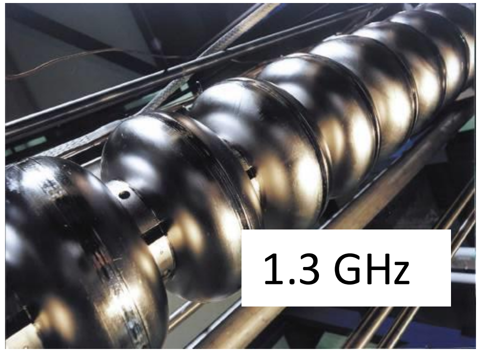


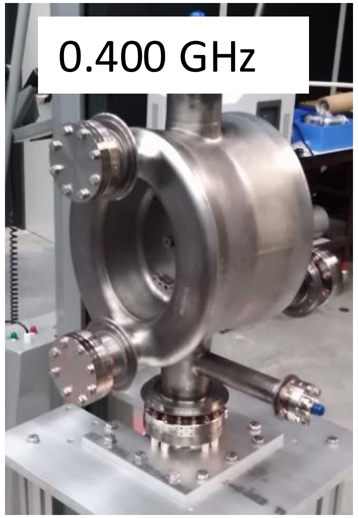


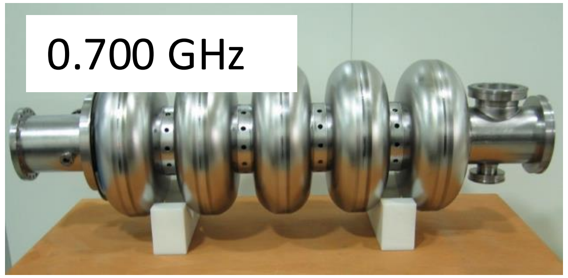


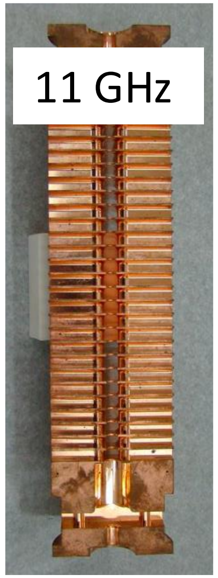
$\lambda \gtrsim d$ resonators



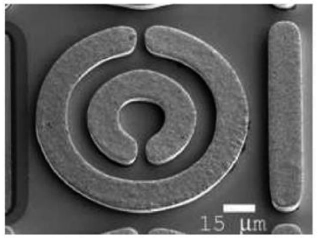


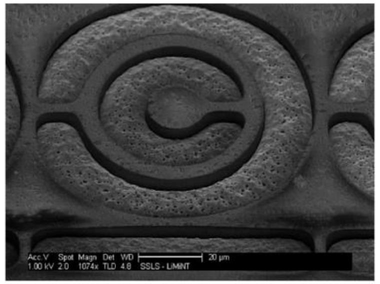


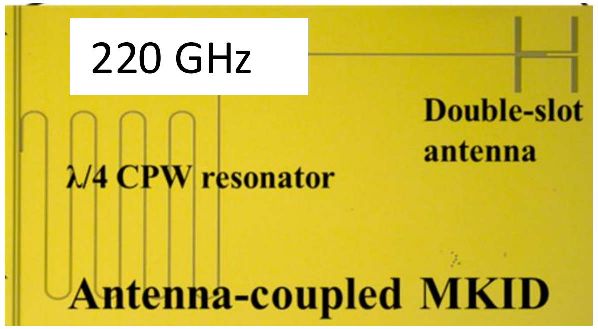




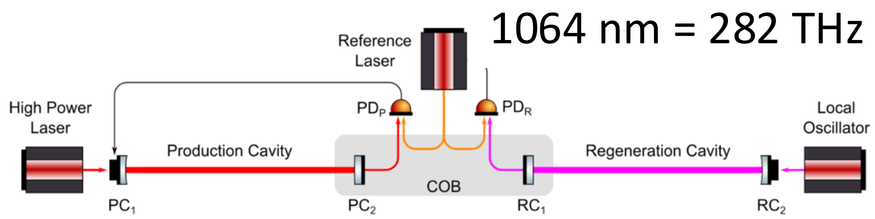
2.7 THz

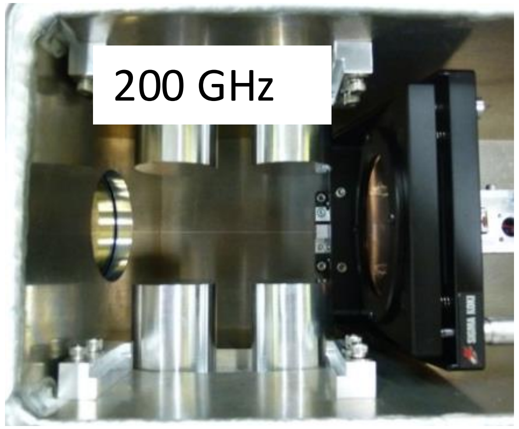


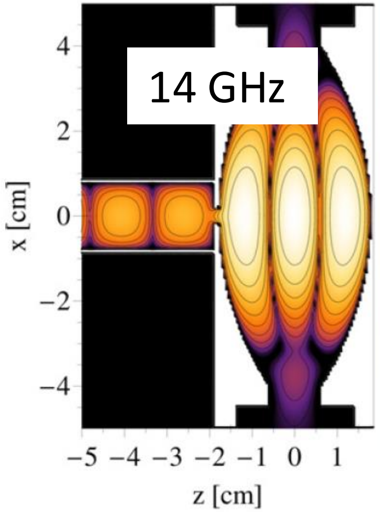


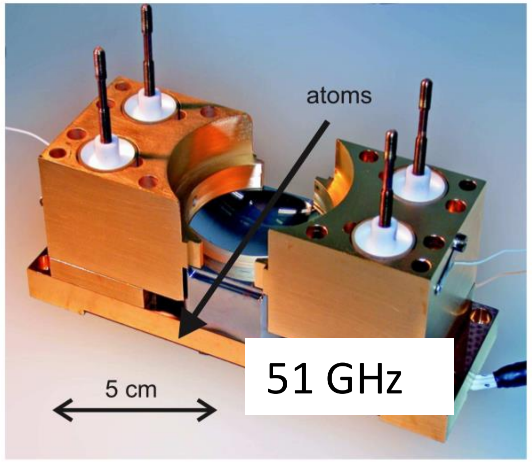


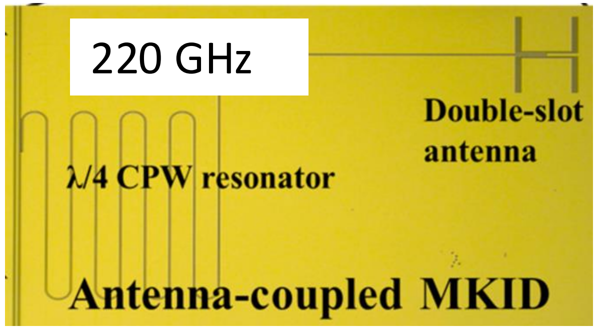
(quasi-)TEM resonators

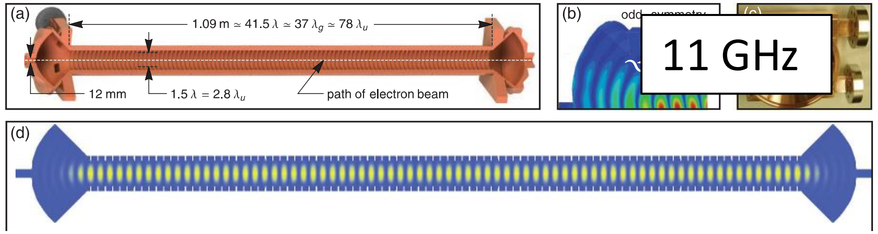




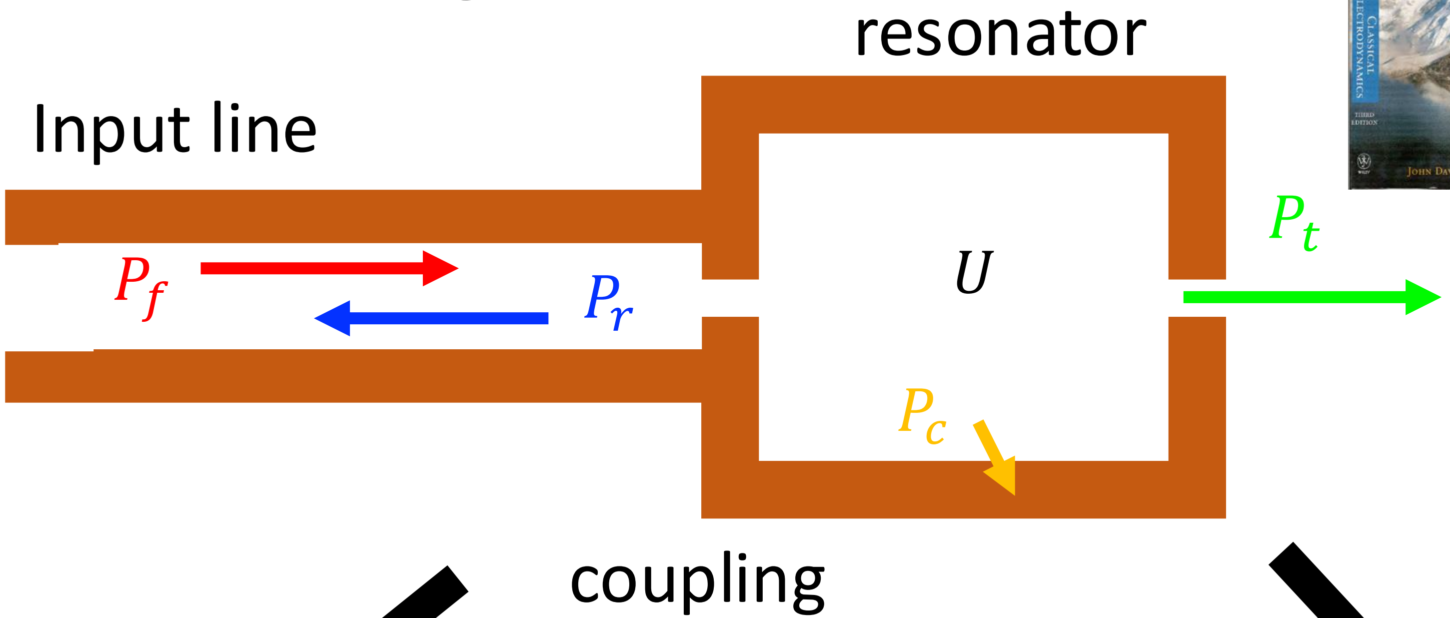


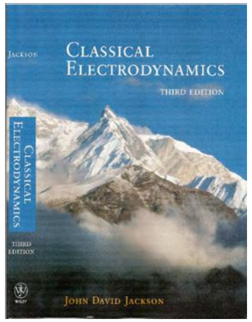


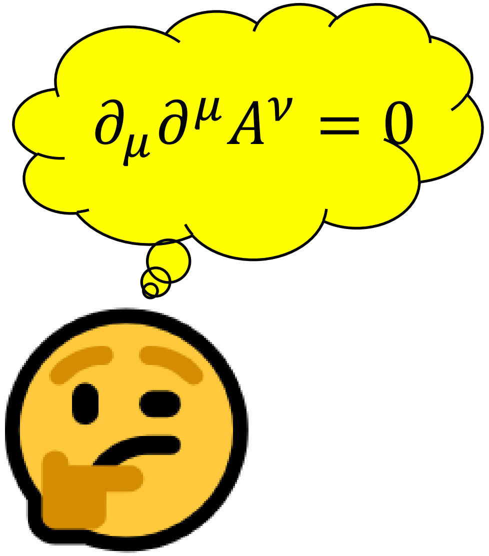


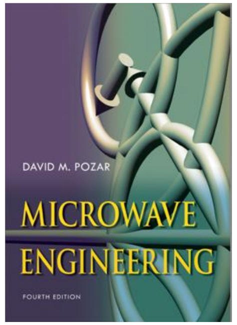


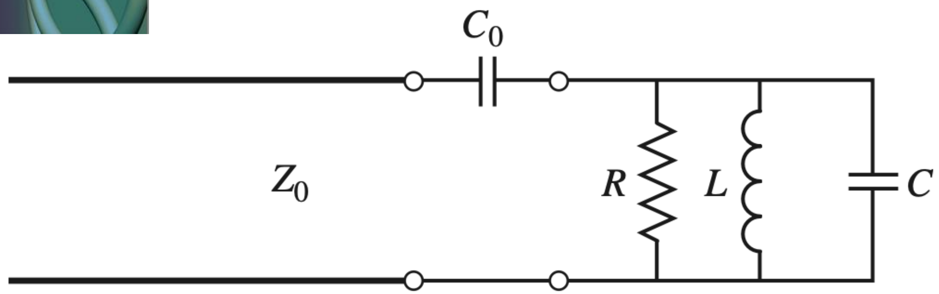
Resonators in general

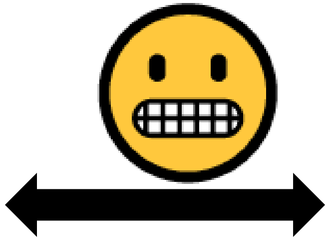


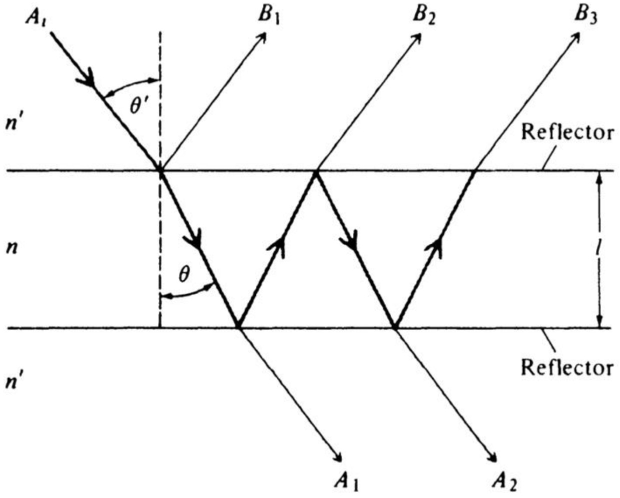


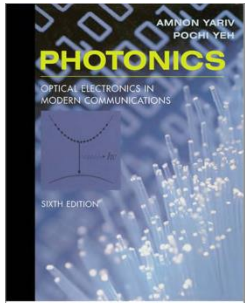




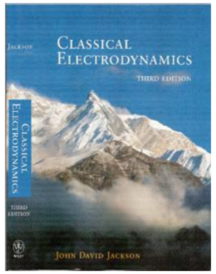






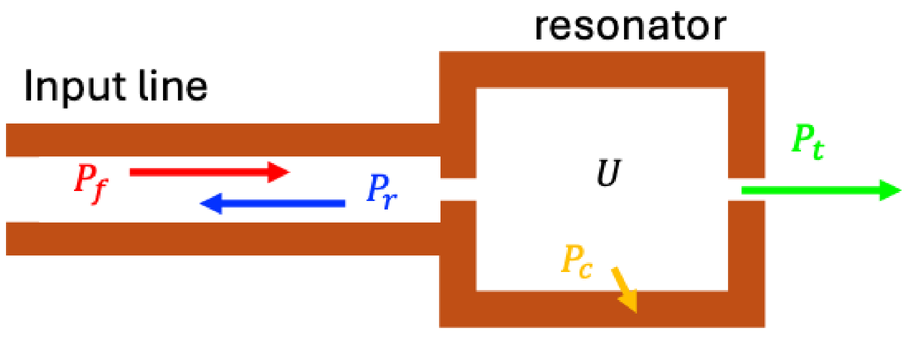


Energy conservation is always valid



The stored energy inside a resonator changes by the balance among in-coming / out-going power

$$\frac{dU}{dt} = P_f - P_r - P_t - P_c$$



Each power is given by integral over port cross section S_j $(j = f, r, t)$

$$P_j = \frac{1}{2}\iint_{S_j} \boldsymbol{E}_j \times \boldsymbol{H}_j^* ds$$

*) Complex conjugate → remark

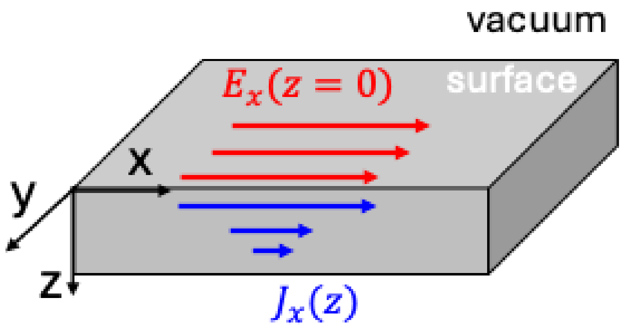
The power dissipation inside the resonator is given by integral over the cavity surface S_c

$$P_c = \frac{1}{2}\iint_{S_c} R_s(\boldsymbol{x})|\boldsymbol{H}(\boldsymbol{x})|^2 ds$$

(Diffraction may be included for open resonators)

With a local surface resistance

$$R_s(\boldsymbol{x}) \equiv \mathrm{Re}\left(\frac{E_x(z=0)}{\int_0^\infty J_x(z)dz}\right) \begin{cases} \sim \sqrt{\dfrac{\pi f \mu_0}{\sigma}} & \text{NC} \\ \sim \dfrac{Af^2}{T}\exp\left(-\dfrac{\Delta}{k_B T}\right) + R_{res} & \text{SC} \end{cases}$$



Quality factors: ratio between energy and power

Each channel of power "leakage" is associated with a parameter

Unloaded quality factor

$$Q_0 = \frac{\omega_0 U}{P_c}$$

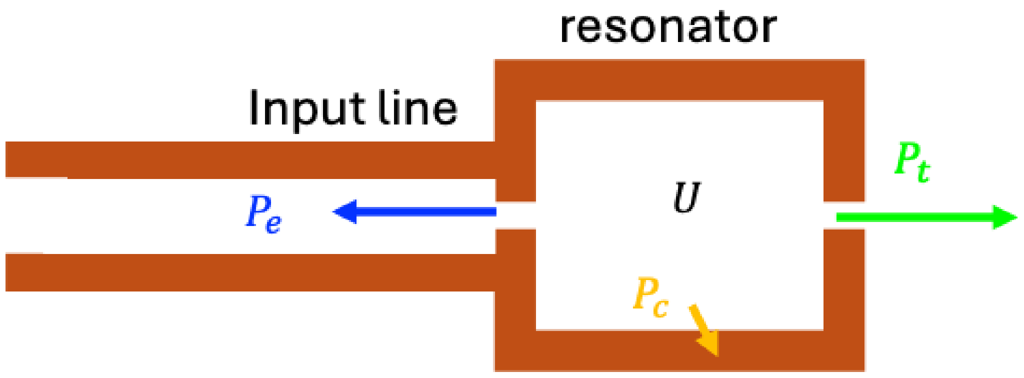
Pick-up probe (transmitted) quality factor

$$Q_t = \frac{\omega_0 U}{P_t}$$

+ diffraction, beam loading, etc

Tricky point: the quality factor of the input line is conventionally called "**external Q**" but

$$Q_{ext} \neq \frac{\omega_0 U}{P_f - P_r} \neq \frac{\omega_0 U}{P_r}$$



Due to the interference between forward and reflected waves

Instead,

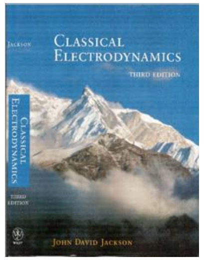
$$Q_{ext} = \frac{\omega_0 U}{P_e}$$

Is defined without the input waves → during "ringing-down" of the stored energy

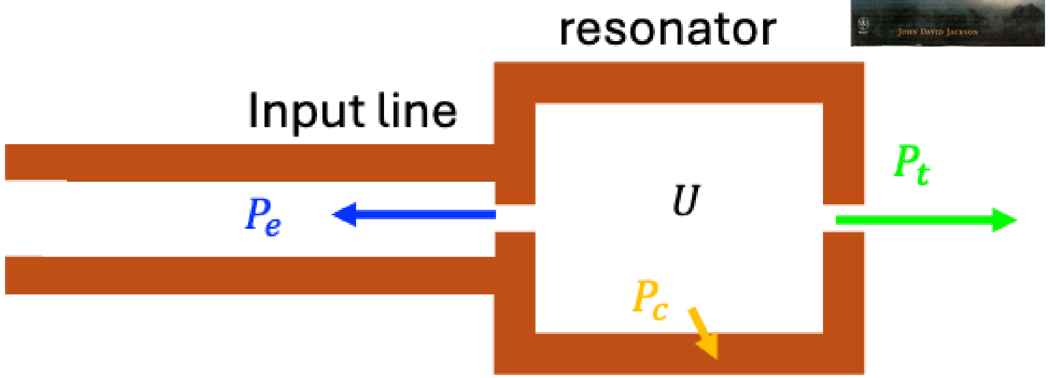
$$P_e \neq P_r \neq P_f - P_r$$

Energy conservation is ignorant of the phases

Still a general theory for any resonators



<u>No input power (switch OFF the input)</u>



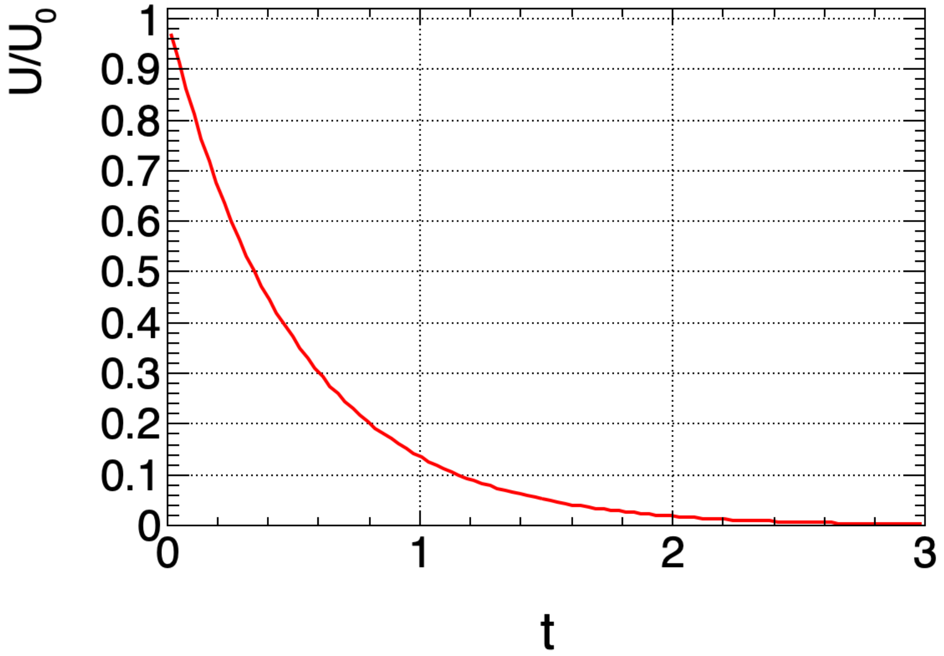
$$\frac{dU}{dt} = \cancel{P_f} - \cancel{P_r} - P_t - P_c$$

$\rightarrow P_e$

$$\frac{dU(t)}{dt} = -\left(\frac{1}{Q_0} + \frac{1}{Q_{ext}} + \frac{1}{Q_t}\right) \times \omega U(t)$$

$\equiv 1/Q_L$ Loaded quality factor Q_L

If Q factors do not depend on the time via energy (i.e. linear response)

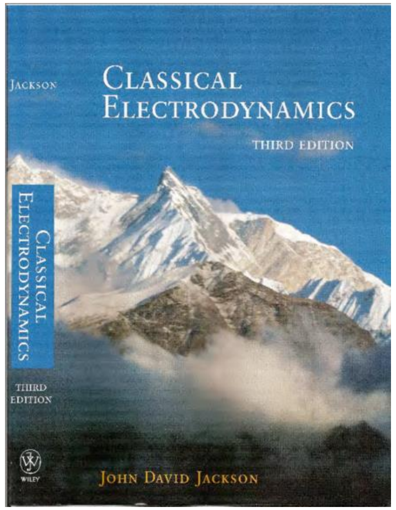


$$U(t) = U_0 \exp\left(-\frac{\omega_0}{Q_L} t\right) \equiv U_0 \exp\left(-\frac{t}{\tau_L}\right)$$

With a time constant $\tau_L = Q_L/\omega_0$ ← beware of $\omega_0 = 2\pi f_0$

U_0: stored energy before switch off

Meaning of Q_L in frequency domain



The microwave inside the resonator is oscillating with $e^{i\omega_0 t}$ and $U \sim |E|^2$ gives

$$U(t) = U_0 \exp\left(-\frac{\omega_0}{Q_L} t\right) \rightarrow E(t) = E_0 \exp\left(-\frac{\omega_0}{2Q_L} t + i\omega_0 t\right)$$

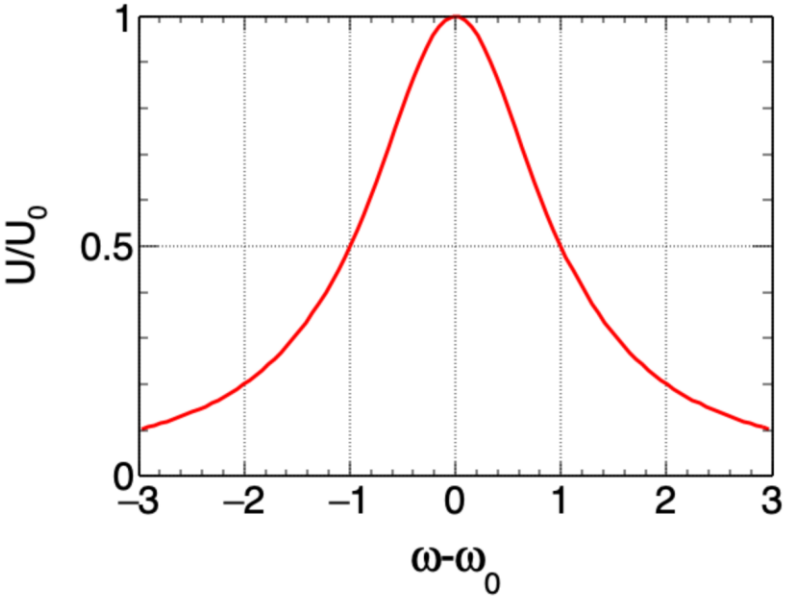
Fourier transform of the field gives

$$\rightarrow \tilde{E}(\omega) = \int_0^{+\infty} E(t') \exp(-i\omega t')\, dt' = E_0 \int_0^{+\infty} \exp\left[\left(-i(\omega - \omega_0) - \frac{\omega_0}{2Q_L}\right) t'\right] dt'$$

$$= \frac{E_0}{i(\omega - \omega_0) + \omega_0/2Q_L} = \frac{-iE_0}{(\omega - \omega_0) - i\,\omega_0/2Q_L}$$

The internal energy in the frequency domain is

$$\tilde{U}(\omega) \sim \tilde{E}(\omega)\tilde{E}^*(\omega) = \frac{|E_0|^2}{(\omega - \omega_0)^2 + (\omega_0/2Q_L)^2}$$



So far, no specific model has been assumed!
(not even Maxwel equation → general nature of linear response system)

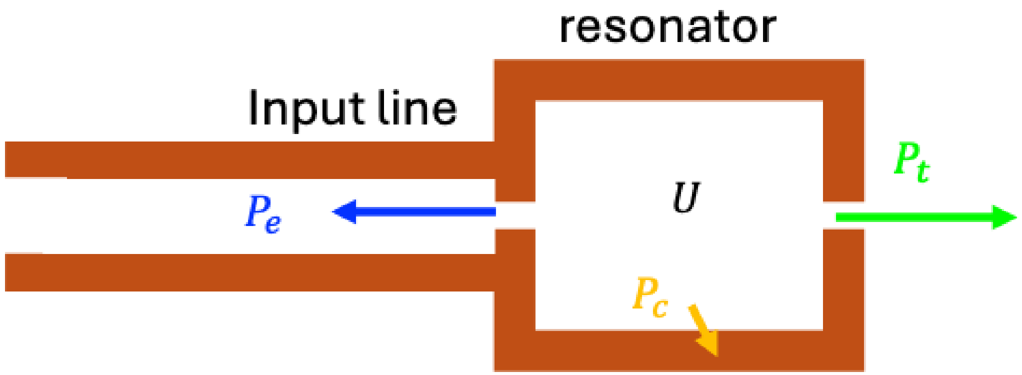
Couling coefficients (still general resonators)

Input coupling

$$\beta \equiv \frac{Q_0}{Q_{ext}}$$

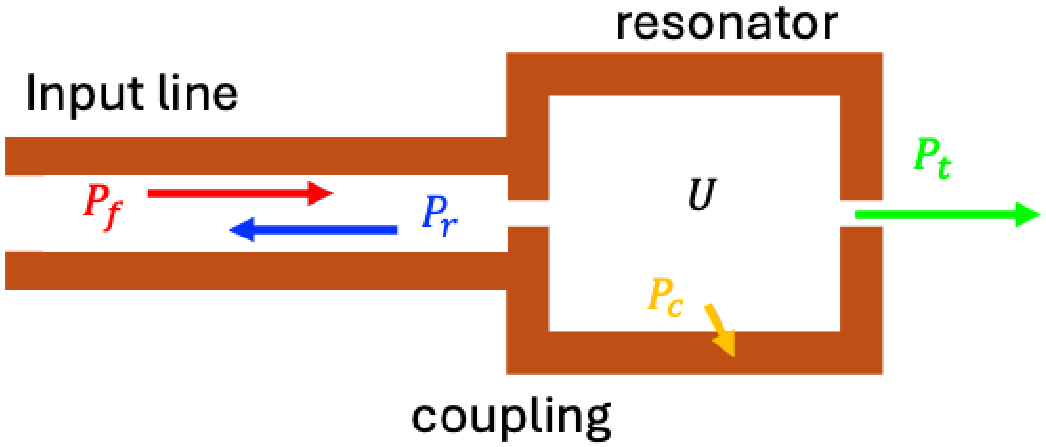
Output coupling

$$\beta_t \equiv \frac{Q_0}{Q_t}$$



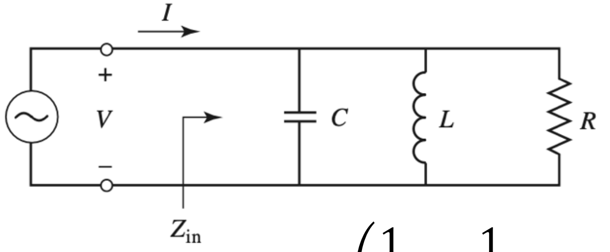
- Over coupling: $\beta > 1 \rightarrow Q_0 > Q_{ext} \rightarrow P_c < P_e$
- Critical coupling: $\beta = 1 \rightarrow Q_0 = Q_{ext} (\therefore Q_L = 2Q_0) \rightarrow P_c = P_e$
- Under coupling: $\beta < 1 \rightarrow Q_0 < Q_{ext} \rightarrow P_c > P_e$

- Then, to my best knowledge, one introduces models to describe specific function of β and relation to P_f and P_r
- If anybody knows more general method, please let me know ☺

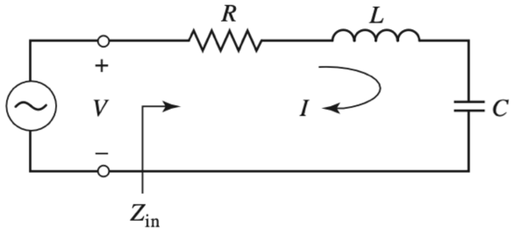


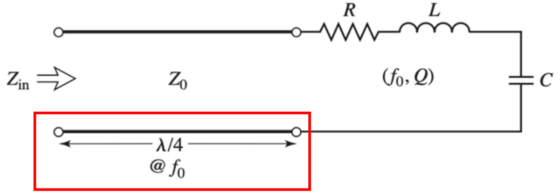
Three methods to tackle more specific aspects

Equivalent circuit model



$$Z_{in} = \left(\frac{1}{R} + \frac{1}{i\omega L} + i\omega C\right)^{-1}$$

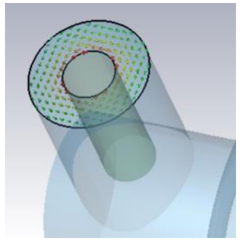




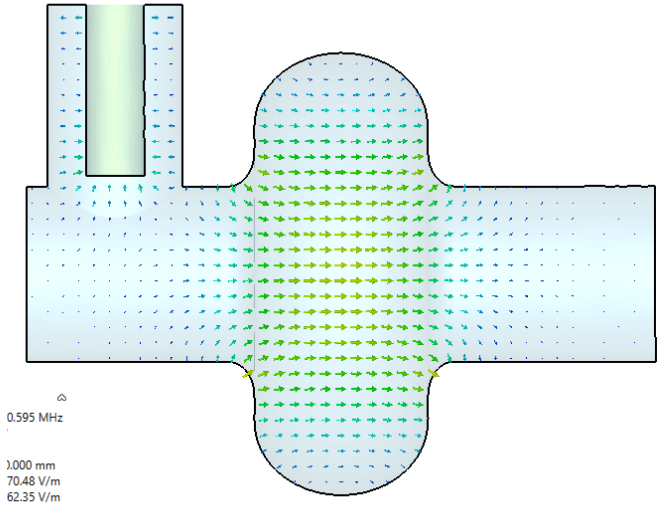
Pros: analytical
Cons: (R, L, C) not obvious, series vs parallel are not well-defined

< 10 GHz

Solve Maxwell equation

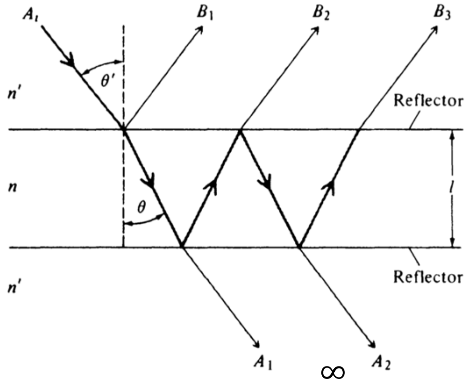


$$Q_{ext} = \frac{\omega_0 U}{P_e} = \frac{\frac{\omega_0}{2\mu_0}\int_V |\boldsymbol{H}|^2 dV}{\frac{1}{2}\int_S \boldsymbol{E} \times \boldsymbol{H} dS}$$

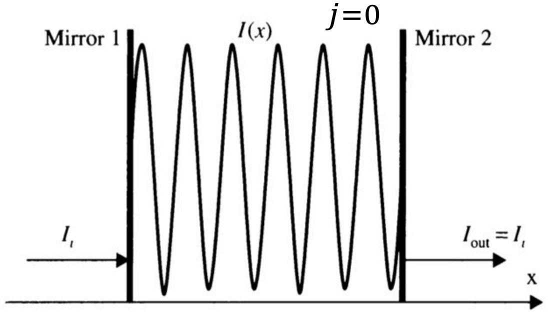


Pros: direct
Cons: Finite Element Method demanding for $\lambda \ll d$ and/or more complicated problems, analytical for only simplest geometry

Interferometer model



$$E_r(z) = t_f E_0 e^{ikz} \sum_{j=0}^{\infty} \left[r_e r_f e^{2ikd}\right]^j$$



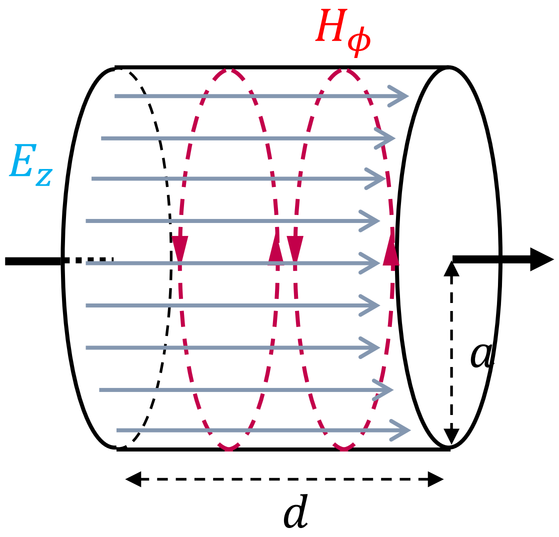
Pros: analytical for $\lambda \ll d$
Cons: quasi-plane waves only diffraction non-predictable

$\gg 10$ GHz

The most typical example: pill-box TM_{010}

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-k_{c,nm}^2\right]\psi(\rho,\phi)=0$$

$$\psi_{nm}(\rho,\phi)=E_0J_m(k_{c,nm}\rho)e^{\pm im\phi} \qquad k_{c,nm}=\frac{j_{nm}}{a}$$



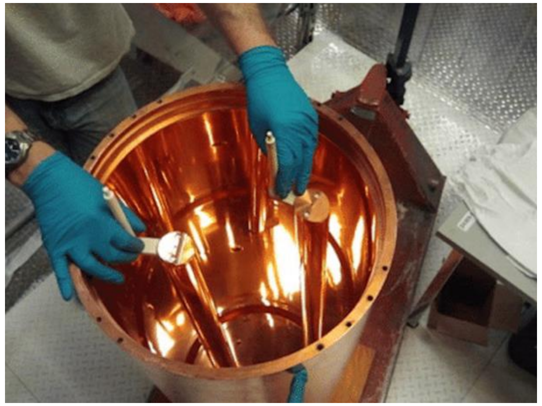
For $(n,m,l)=(0,1,0)$

$$\omega_{010}=\frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{j_{01}}{a}\right)^2+\left(\frac{0\cdot\pi}{d}\right)^2}\sim\frac{2.405}{a\sqrt{\mu\varepsilon}}\sim\frac{2.405}{a/c}$$

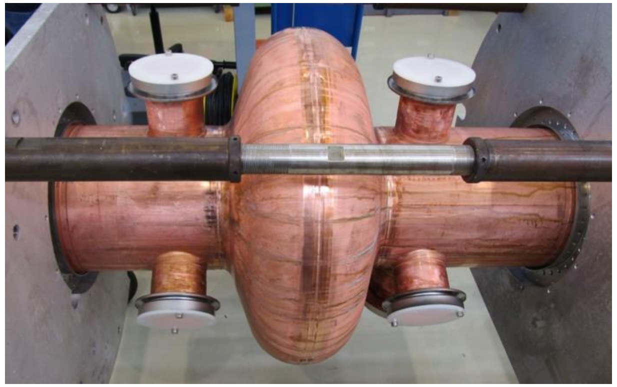
- Baseline of the most typically used accelerating cavities & axion detectors
- Curvature to further optimize efficiency

$$\begin{cases} E_z(\rho,\phi,z,t)=E_0J_0\left(\frac{2.405\rho}{a}\right)e^{-i\omega t} \\ H_\phi=-i\sqrt{\frac{\varepsilon}{\mu}}E_0J_1\left(\frac{2.405\rho}{a}\right)e^{-i\omega t} \\ E_\rho=E_\phi=H_\rho=H_z=0 \end{cases}$$

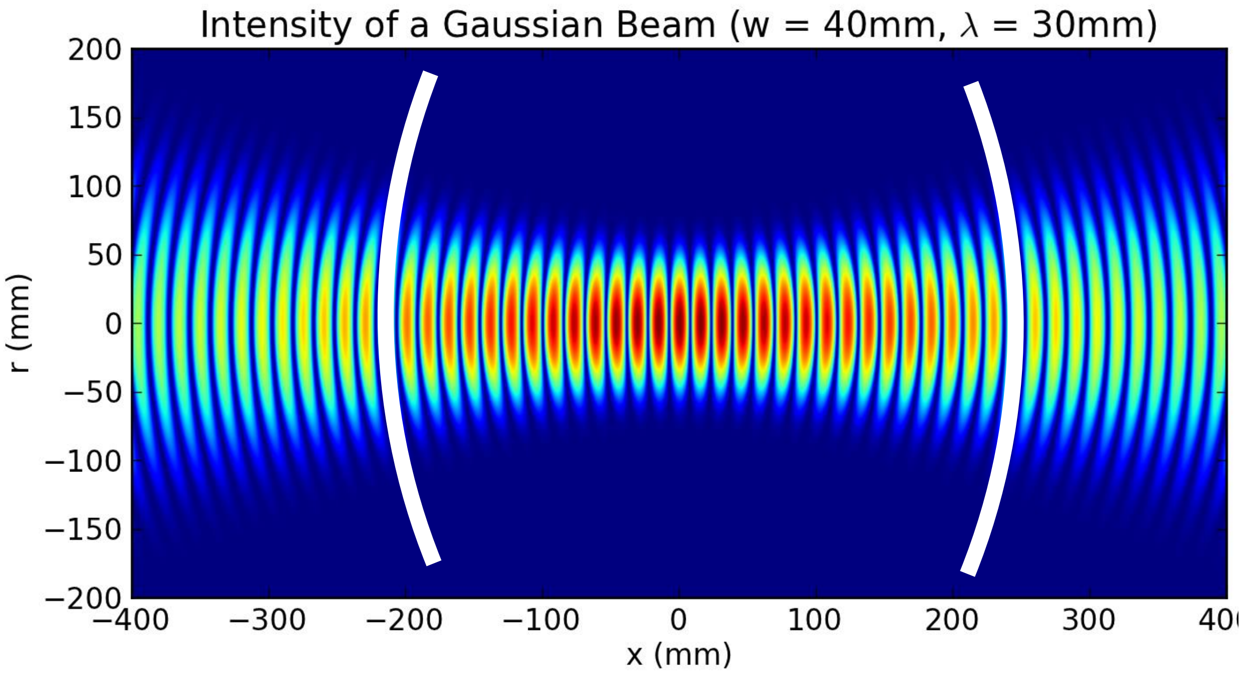
ADMX cavity

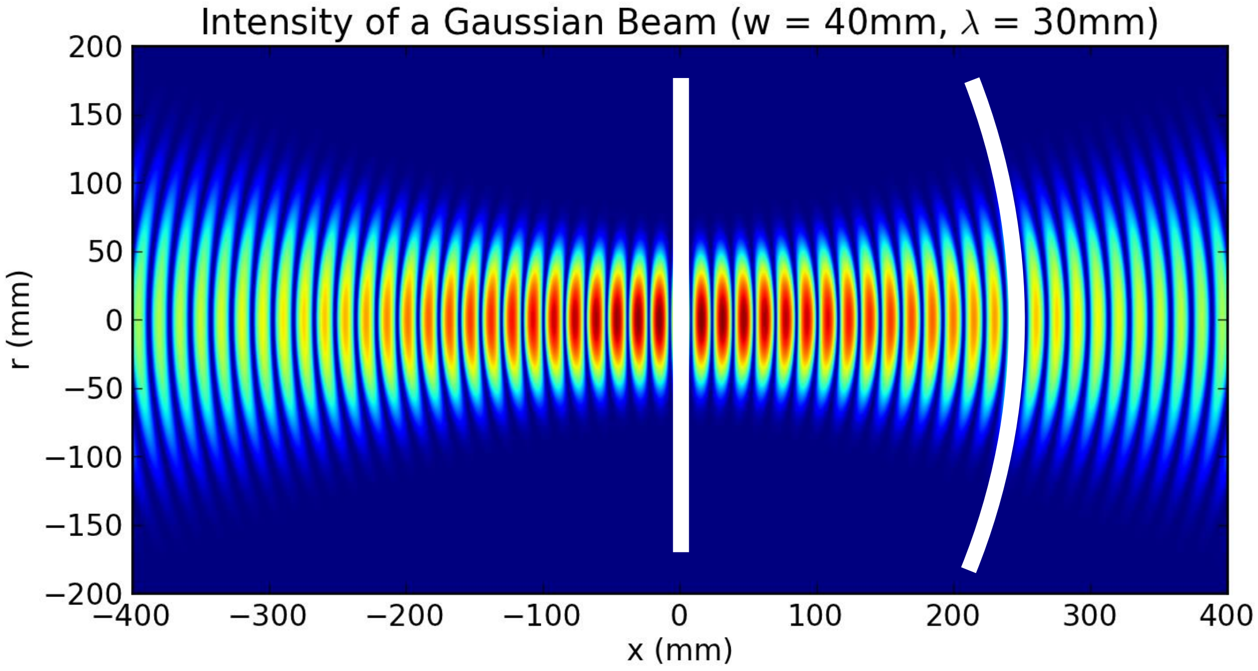


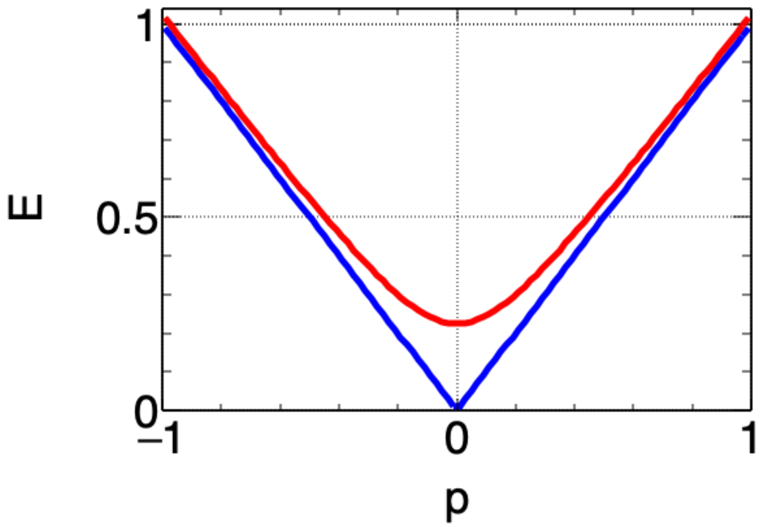
LHC cavity



Fabry-Pérot resonators $R(z) = z\left[1 + \left(\frac{\pi w_0^2}{\lambda z}\right)^2\right]$





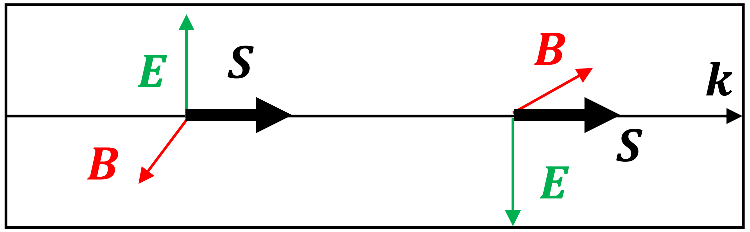


- Microwaves in Fabry-Pérot resonator couple to relativistic axions / dark photons → LSW
- For dark matter axions, dielectric disks to cancel the negative polarity (MADMAX) → Part 1

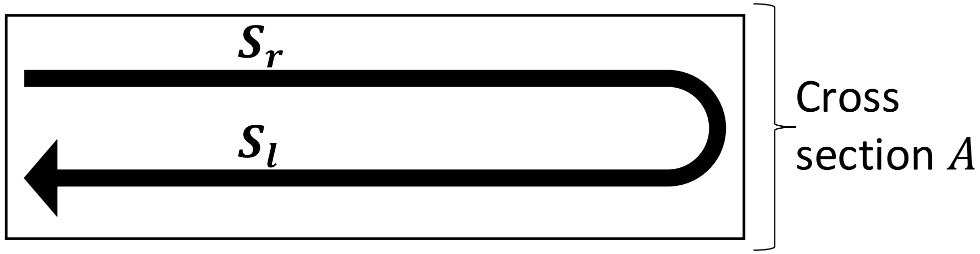
Remark: circulating power instead of stored energy

Some cavities / resonators have TEM or linearly polarized modes as their eigenmodes

TEM: electric and magnetic fields are perpendicular to an axis
→ Poynting vector $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}$ can be locally defined
→ Time averaged $\boldsymbol{S}$ over $1/\omega$ is still 0 so that no net power flow



Although the standing-wave does not have net power flow but has stores energy, it can also be interpreted as a superposition of left- and right- bound waves
→ ***Photons*** bounce left and right



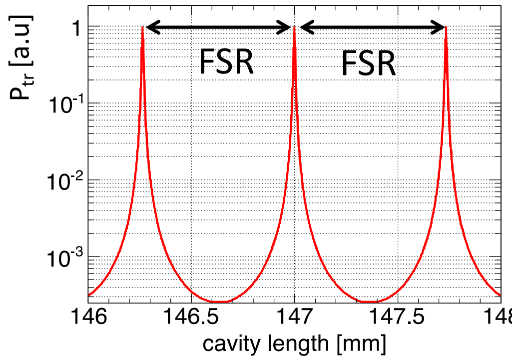
In case of critically coupled cavity, the circulating power is defined via finesse $F = FSR/FWHM$

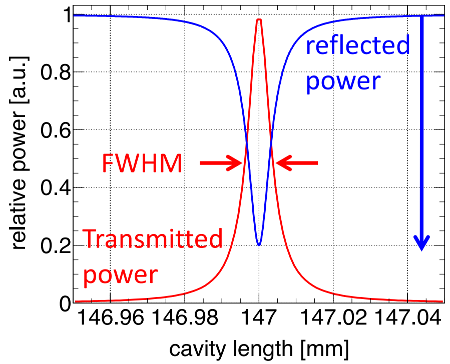
$$P_{cy} = \frac{F}{\pi} P_f$$

Net photon flux is proportional to the circulating power

$$I = \frac{P_{cy}}{\hbar\omega A} = \frac{F}{\pi\hbar\omega A} P_f \propto E^2, B^2, E \times B$$

→ The circulating power is often used in Fabry-Pérot resonators

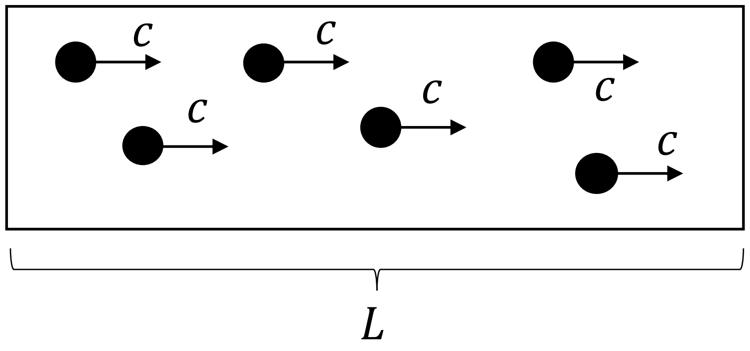




Internal energy vs circulating power

The total number of photons can be expressed by the photon flux

$$N = I \times \frac{L}{c} \times A$$



The finesse of the resonator (in critical coupling)

$$F = \frac{Q_0}{2 \times \#\text{ of nodes}} = \frac{Q_0}{4L/\lambda} = \frac{c}{4Lf} Q_0 = \frac{\pi c}{2L\omega} Q_0$$

$$I = \frac{P_{cy}}{\hbar\omega A} = \frac{F}{\pi\hbar\omega A} P_f$$

Photon density

$$n = \frac{N}{V} = \frac{U}{V\hbar\omega} = \frac{1}{V\hbar\omega}\frac{Q_0 P_f}{\omega} = \frac{1}{V\hbar\omega}\frac{P_f}{\omega}\frac{2L\omega}{\pi c} F = \frac{2}{c}\frac{F}{\pi\hbar\omega A} P_f = \frac{2}{c}\frac{P_{cy}}{\hbar\omega A}$$

$$\rightarrow U = \frac{2L}{c} P_{cy}$$

Both internal energy and circulating power can be used to characterize the performance of cavities

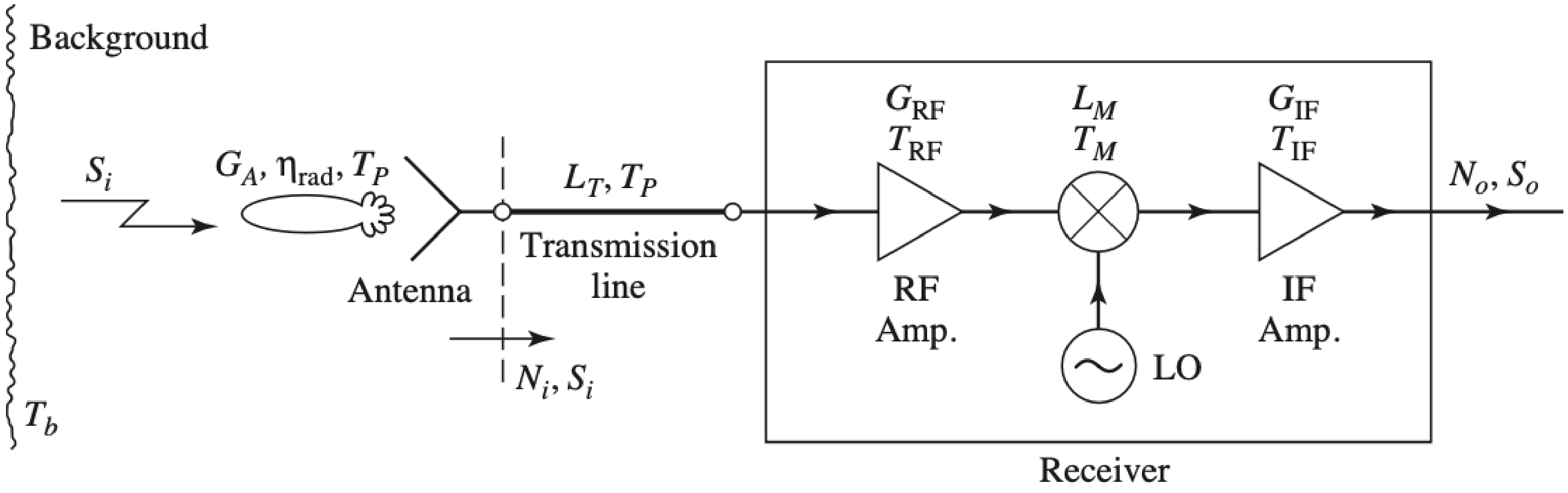
- For over-size cavities ($L \gg \lambda$, eg Fabry-Pérot resonator), P_{cy} is useful and F is a figure of merit
    - The internal energy and Q_0 are increased by longer L and often misleading
    - Spectroscopy (quantum transition is proportional to energy density, not total energy)
    - Sensitivity of axion experiments with FP resonators is proportional to photon flux

Part 2: classical detection scheme

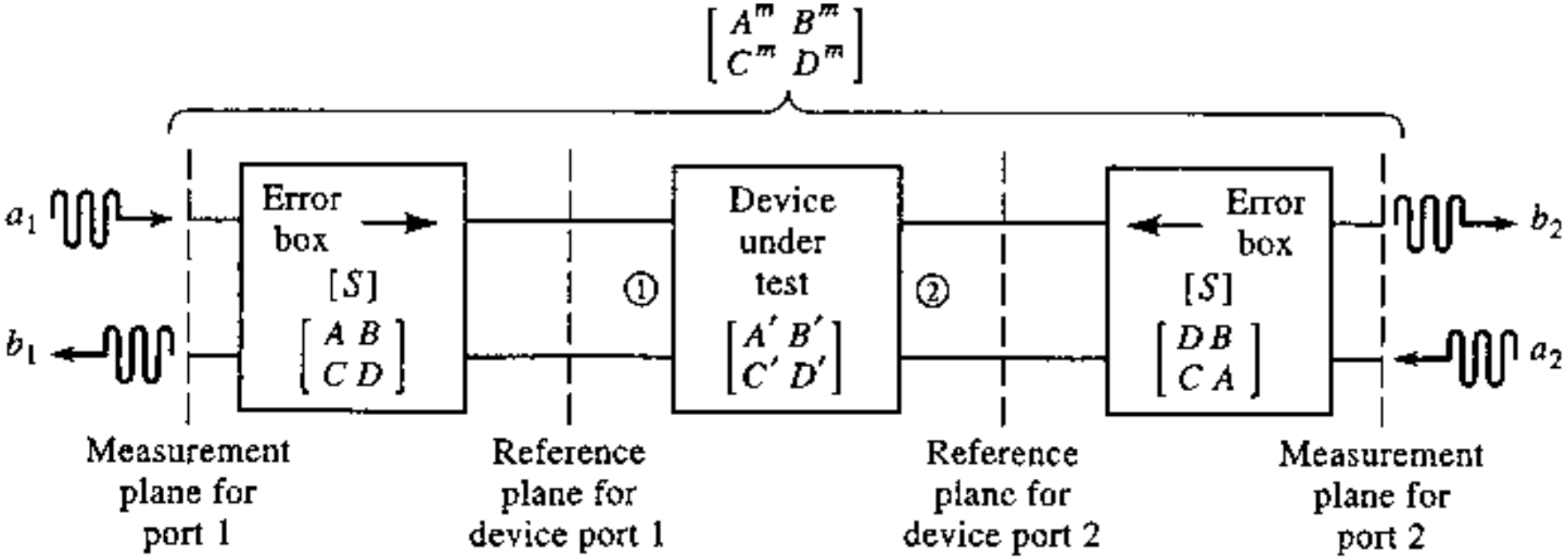
- Boundary conditions
  - Normal conductors
  - Superconductors
  - Dielectric materials: insulator and semi-conductors
- Microwave resonators
  - Waveguide and transmission line
  - Resonant cavity
  - Fabry-Pérot resonator
- Analog and digital system
  - Amplifier, circulator, mixer & analogue down-conversion
  - I/Q sampling & digital down-conversion
- Data processing and noise
  - FFT: coherent and incoherent integral
  - Narrowband and broadband
  - Thermal noise and standard quantum limit
- Conclusion of part 2

Typical RF measurement setups

Passive measurement

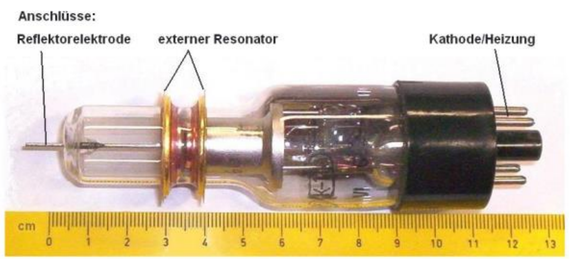


Active measurement

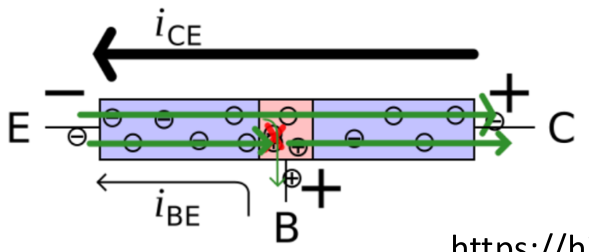


Linear amplifiers

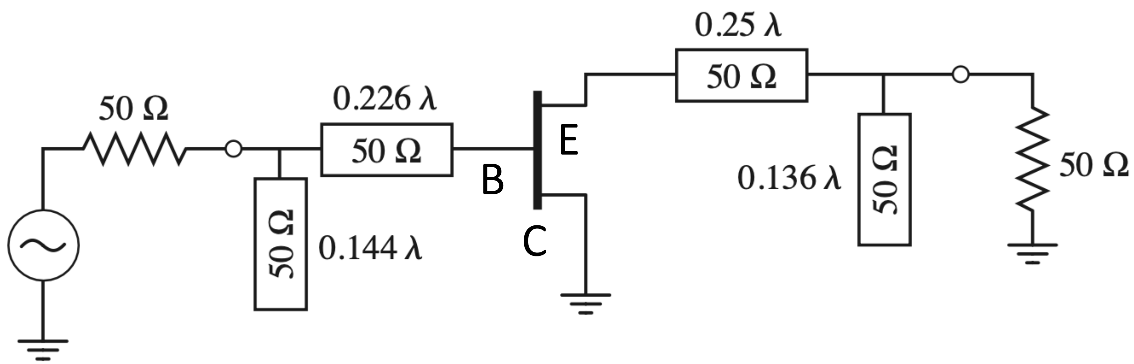
From wiki

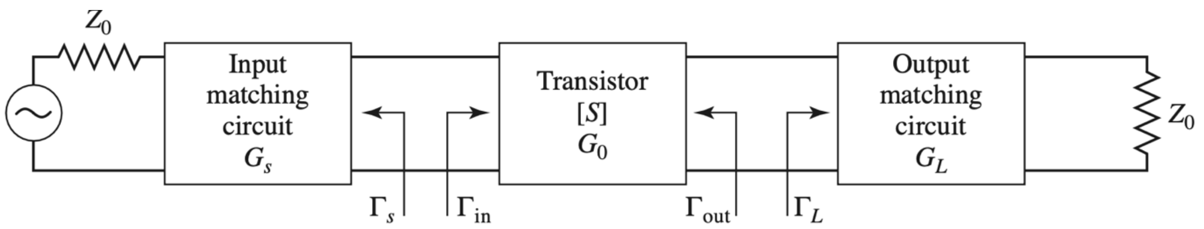


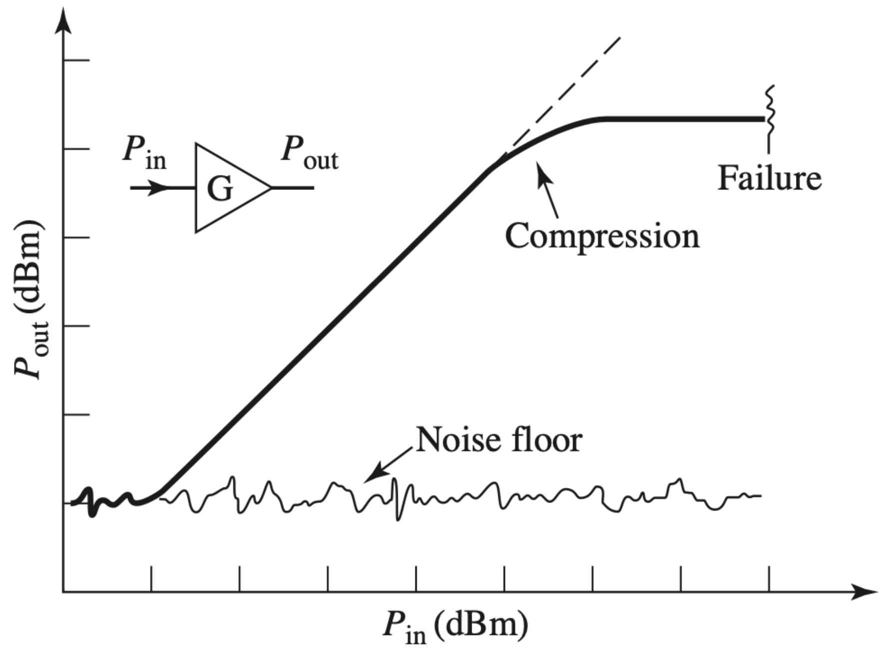
- High power amplifier (**vacuum tube** or combined transistors)
  - When one needs high power / fields for applications (cf accelerator cavity)
- Low-noise amplifier (**transistor**)
  - When the signal level is below the noise floor of the detector (power sensor or ADC)
  - Tunnelling effect in solid state → amplification
- Users do not have to care transistors
  - Input & output matching are more crucial
- MADMAX "LNF optimize S/N at the cost of input mismatch"
  - We usually add a circulator in front of an amplifier



https://hilelectronic.com/low-noise-amplifiers/

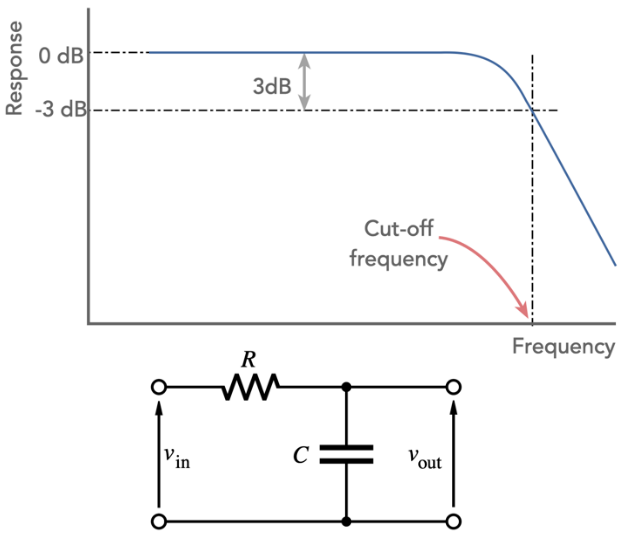




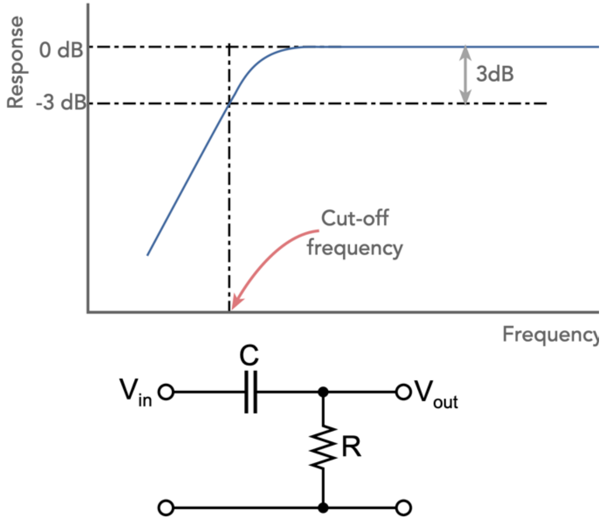


Analogue filters: four types

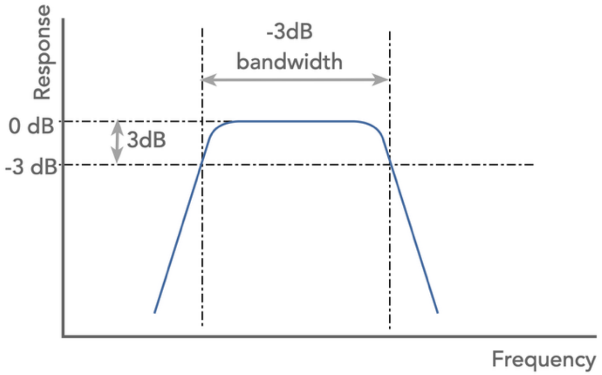
Low-pass filter



high-pass filter

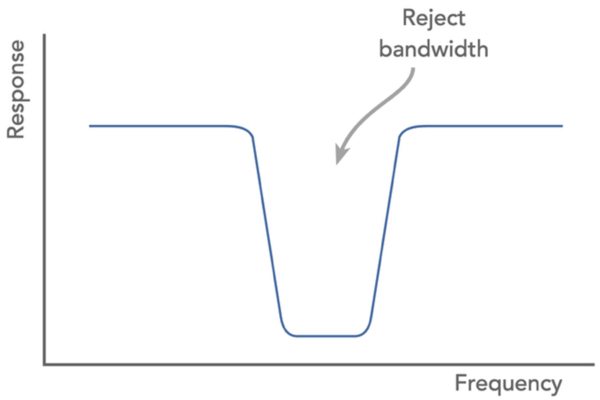


band-pass filter



→ Narrow band
→ high-Q resonator

band-rejection filter



→ Narrow band
→ Notch filter

- They are essential to remove parasitic MW and/or noise outside the interesting signal bands
- Amplifiers are broadband → to be combined with filters to avoid reduce broadband noise amplification

Circulator / isolator

Ferrite ∈ ferrimagnetic ≠ ferromagnetic

Polarization dependence
→ direction dependence

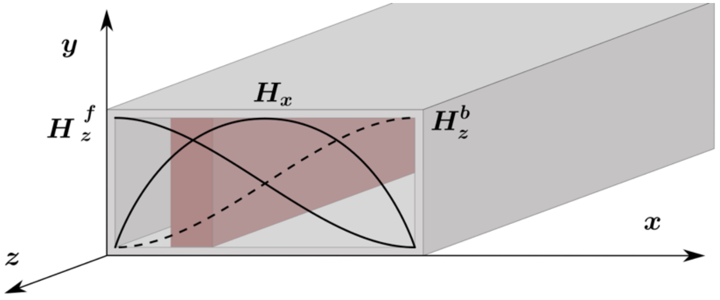
$$\begin{cases} \nabla \times \boldsymbol{E} = -i\omega\overleftrightarrow{\mu}\boldsymbol{H} \\ \nabla \times \boldsymbol{H} = i\omega\varepsilon\boldsymbol{E} \end{cases}$$

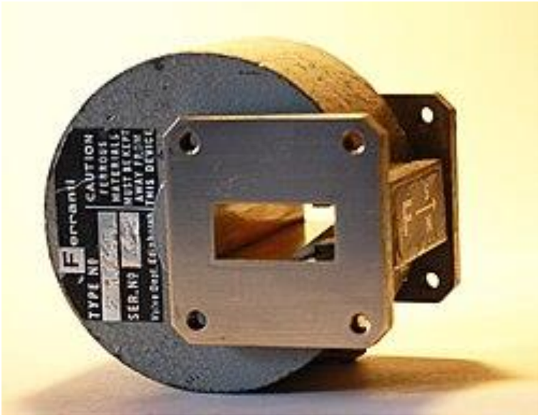
$$\overleftrightarrow{\mu} = \begin{bmatrix} \mu & 0 & i\kappa \\ 0 & \mu_0 & 0 \\ i\kappa & 0 & \mu \end{bmatrix} \qquad \mu = \mu_0\left(1 + \frac{\omega_0\omega_m}{\omega_0^2 - \omega^2}\right) \qquad \kappa = \mu_0\frac{\omega\omega_m}{\omega_0^2 - \omega^2}$$

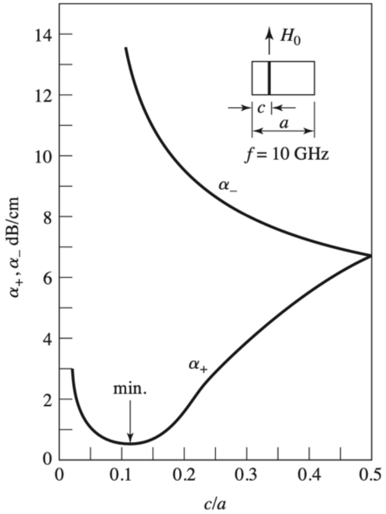
ω_0: Larmor presession $\quad \omega_m \propto$saturation magnetization

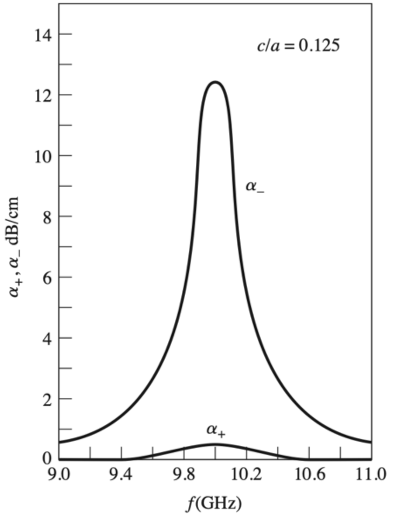
From wiki

$$[S] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$



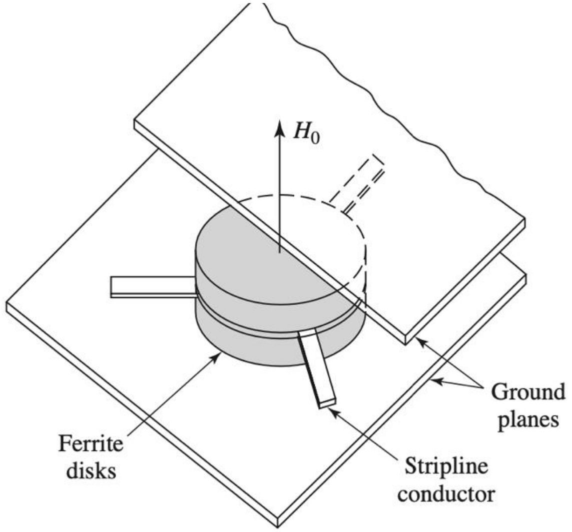


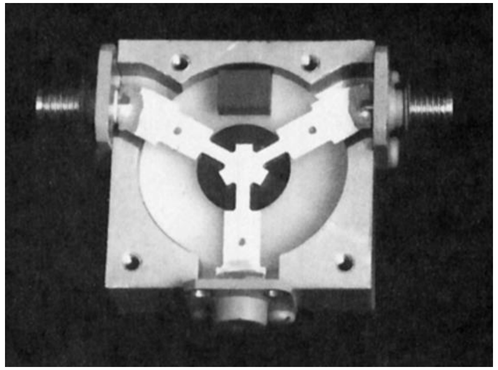


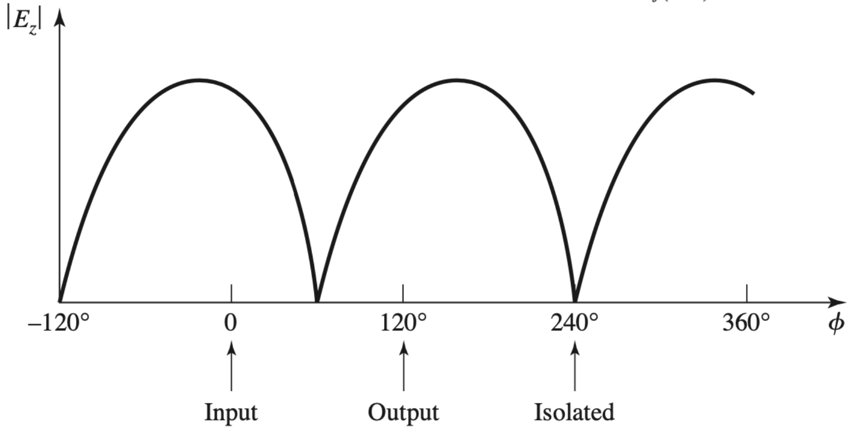


Narrow band = BP

$$[S] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

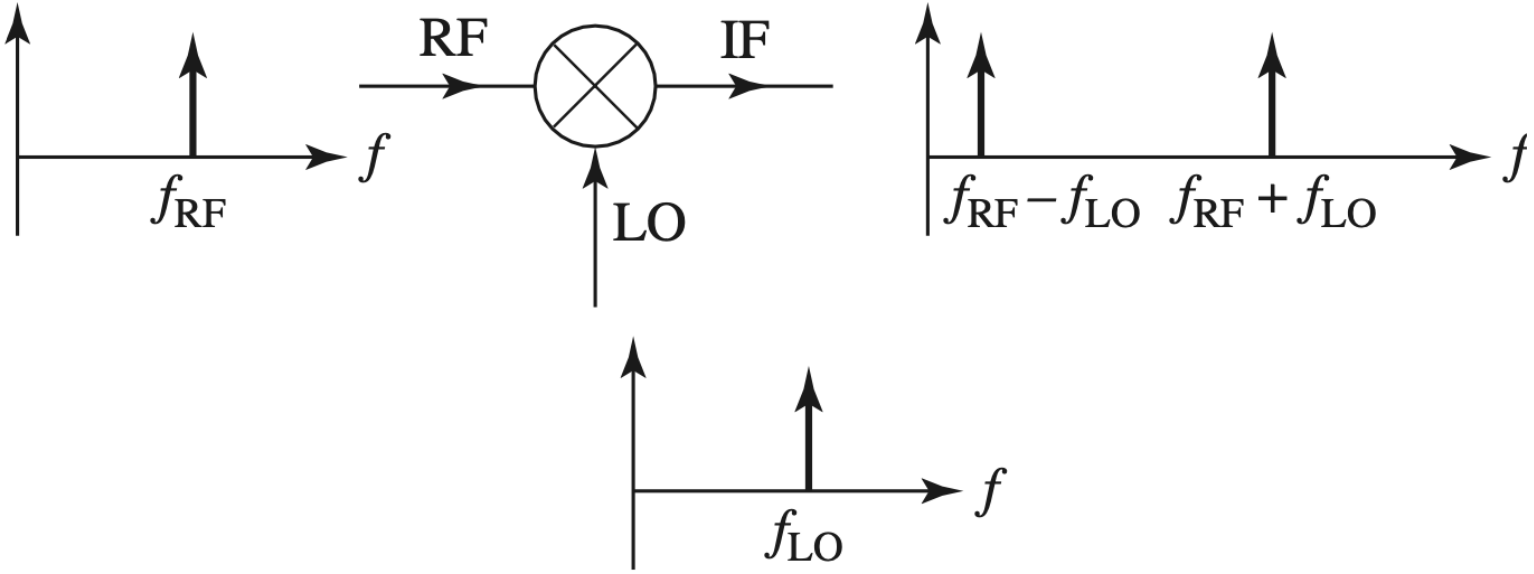


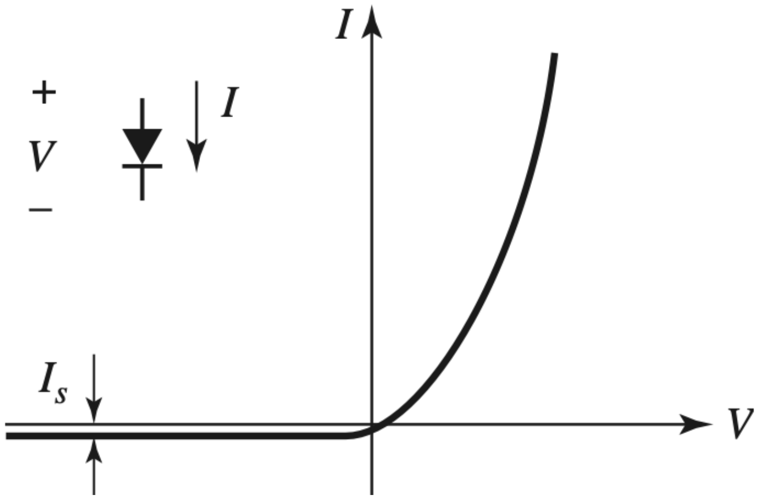




→ Typically add them before and/or after amplifiers to avoid harmful standing waves

Diode mixer (≠ tensor product)





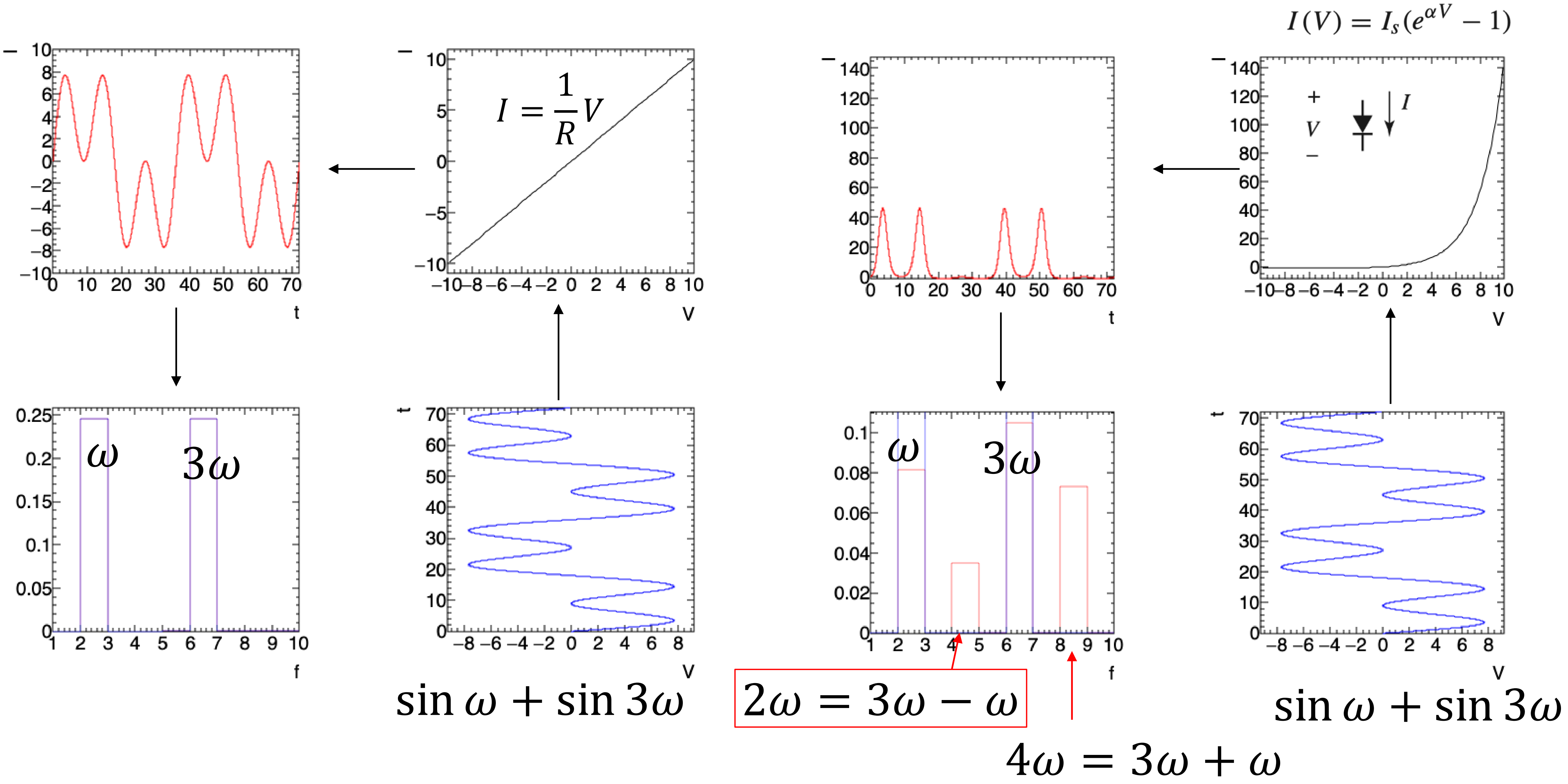
$$I(V) = I_s(e^{\alpha V} - 1)$$

This is the **core** of modern **low-noise** microwave engineering

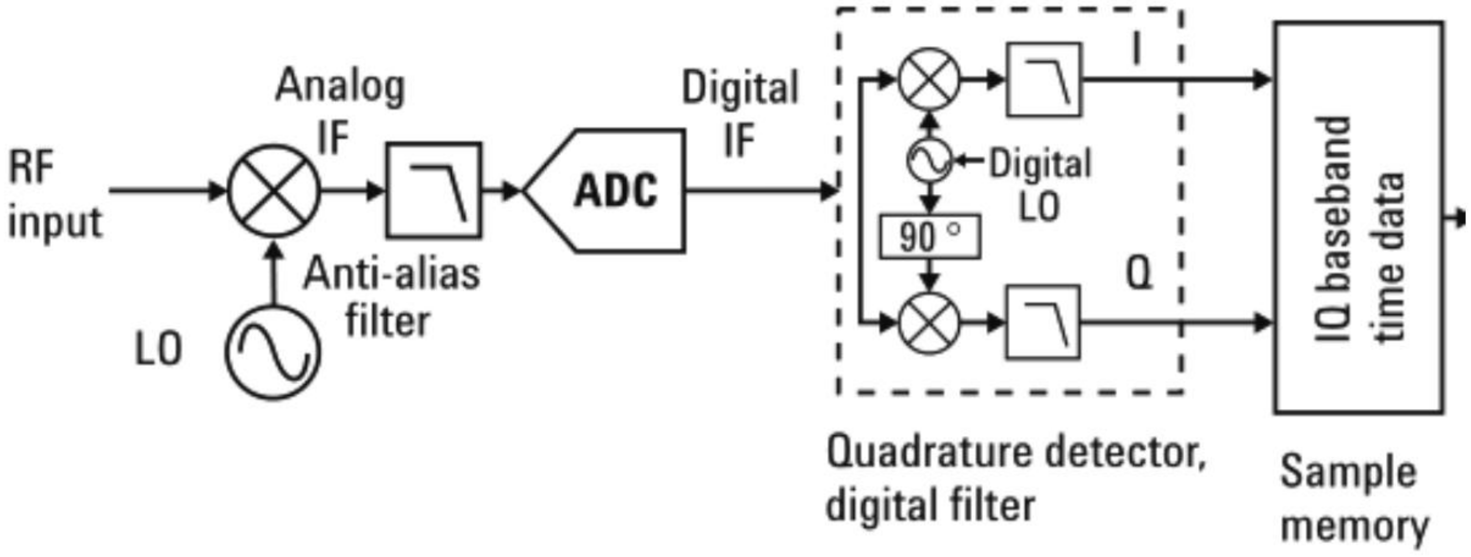
→ Superheterodyne (analogue + digital)

→ Phase and frequency information is kept

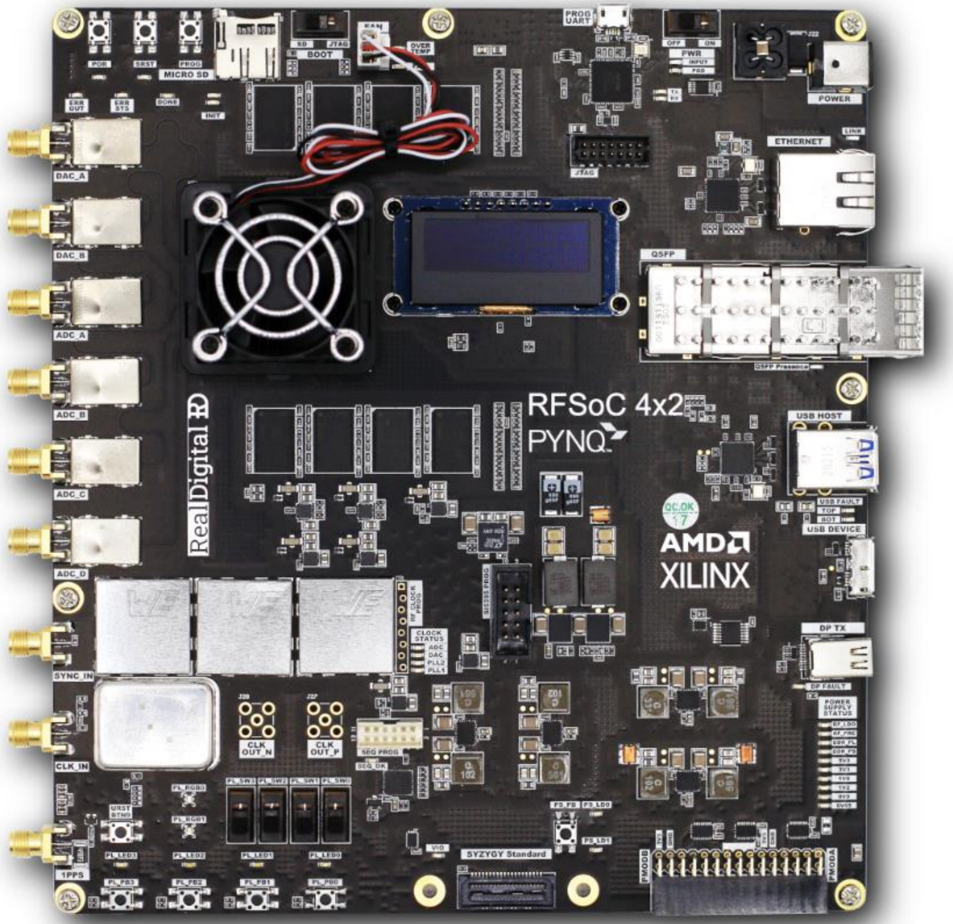
Heterodyne via mixer: analogue down-conversion



Digital processing: FPGA after ADC

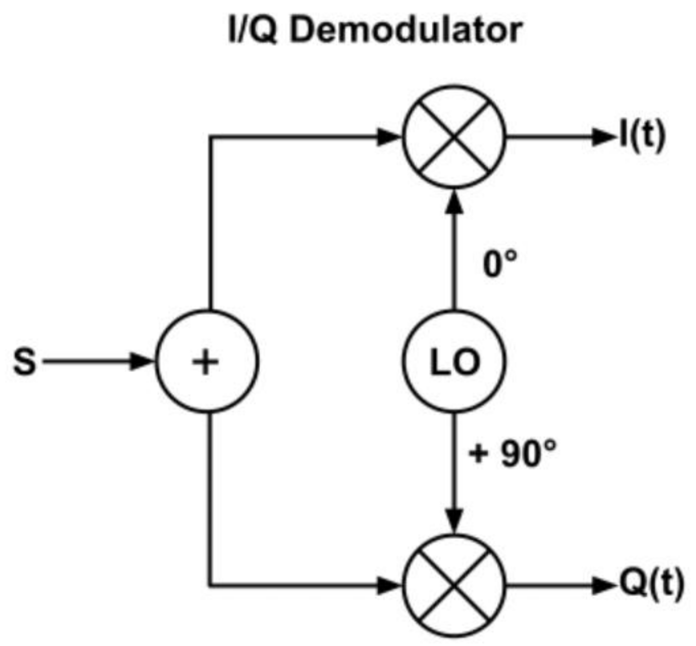


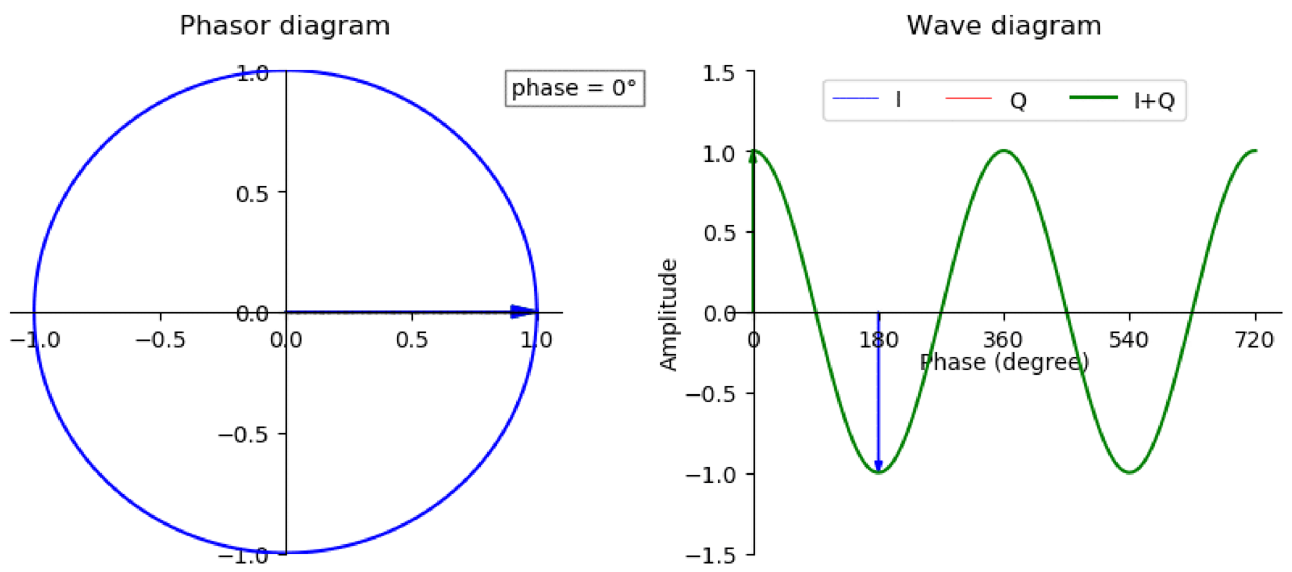
RealDigital RFSoc 4x2



R&S FSW43 real-time spectrum analyzer

in-phase (I) and quadrature (Q) sampling

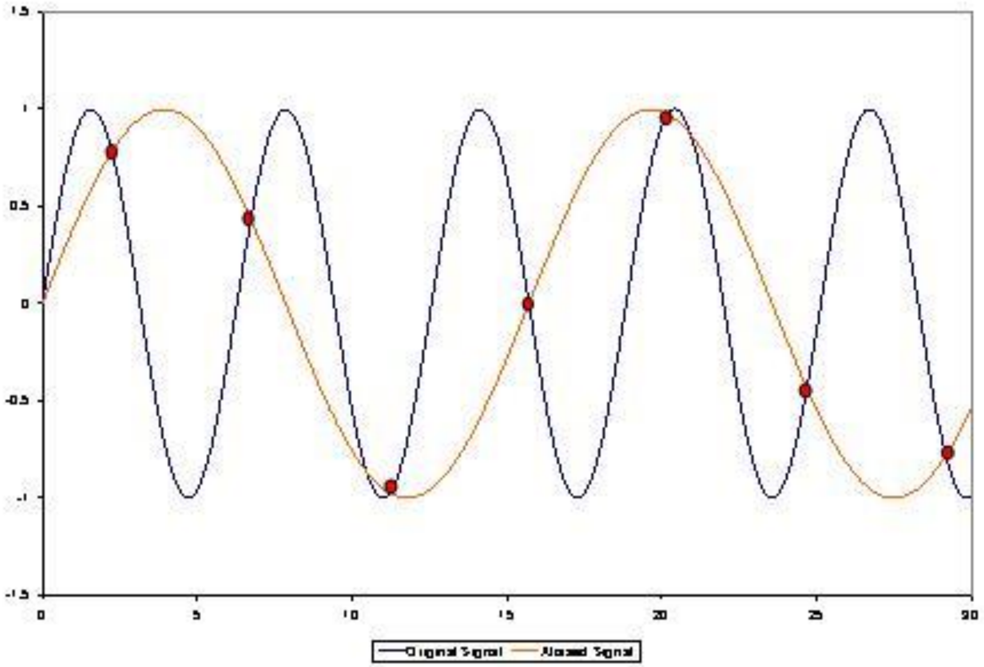


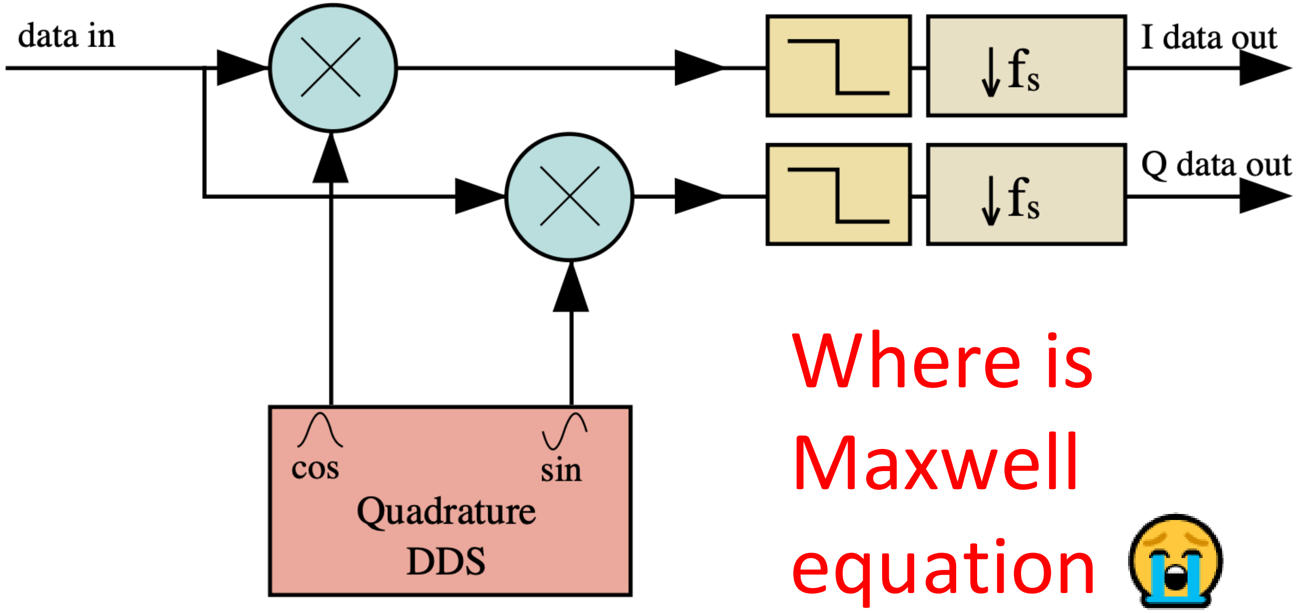


$$RF(t) = I(t)\cos(\omega t) + Q(t)\sin(\omega t)$$

- After sufficient analogue superheterodyne, one can directly digitize RF signal
- Standard of modern microwave engineering
- The channel reading of ADC is linear to input voltage
- Raw data to process is digitized $I(t)$ and $Q(t)$ in modern microwave system

Digital down conversion (**decimation in FPGA**)





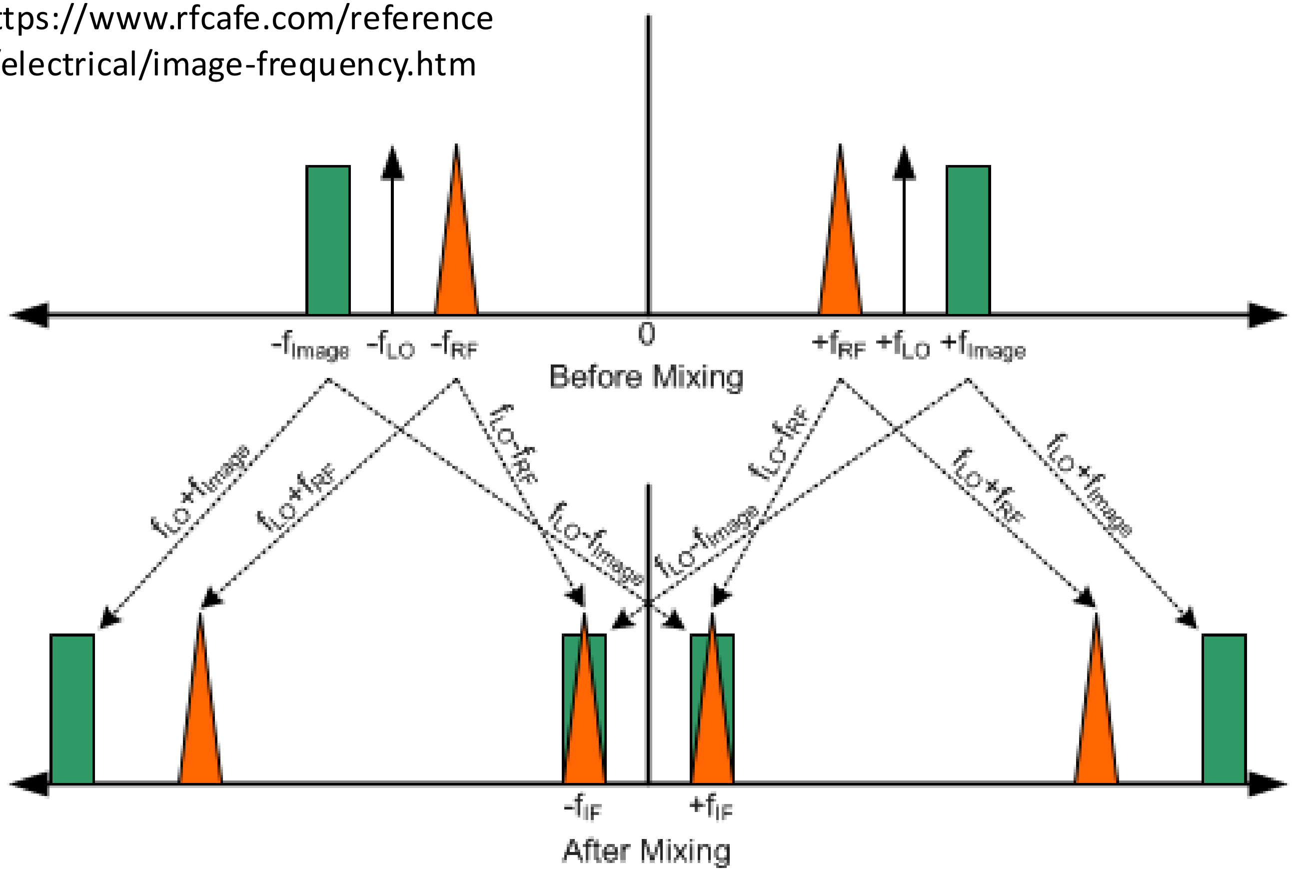
Where is Maxwell equation 😭

- Make use of Nyquist theorem
  - A periodic signal must be sampled (sampling freuqneyc ω_s) at more than twice the highest frequency component of the signal → otherwise aliasing
- Under-sampling on purpose
  - Aliasing = frequency down conversion $\omega_{LO} = \omega_s$ → $\omega_{IF} = \omega_{RF} - \omega_s$
- Typically implemented in FPGA together with digital filters in modern microwave system
- One can flexibly control the downconversion by decimation factor unlike analogue heterodyne

Digital filters to remove image problems

Where is Maxwell equation 😭

https://www.rfcafe.com/references/electrical/image-frequency.htm



- We usually use analogue filters with analogue mixers
- digital down-conversion is associated with digital filters in FPGA
- This is one of the reasons why modern digital processing of microwaves is like a black box for physicists

There are also spurious peaks generally in the microwaves (eg amplifier's nonlinearity, etc)
→ digital filter implementation + down conversion seems like key of FPGA code development

Part 2: classical detection scheme

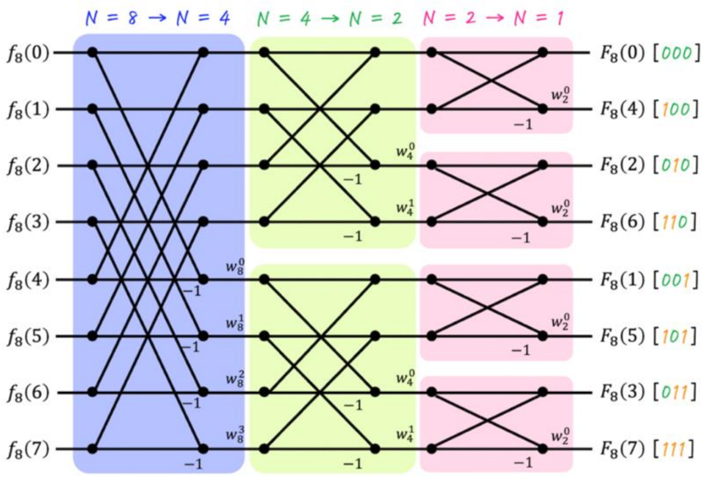
- Boundary conditions
  - Normal conductors
  - Superconductors
  - Dielectric materials: insulator and semi-conductors
- Microwave resonators
  - Waveguide and transmission line
  - Resonant cavity
  - Fabry-Pérot resonator
- Analog and digital system
  - Amplifier, circulator, mixer & analogue down-conversion
  - I/Q sampling & digital down-conversion
- Data processing and noise
  - FFT: coherent and incoherent integral
  - Narrowband and broadband
  - Thermal noise and standard quantum limit
- Conclusion of part 2

Discrete Fourier Transform (DFT) and Fast Fourier Transform (FFT)

$$G(f) \equiv \sum_{t=0}^{N-1} g(t) \exp\left(-i\frac{2\pi f t}{N}\right)$$

$$g(t) \equiv \frac{1}{N}\sum_{f=0}^{N-1} G(f) \exp\left(i\frac{2\pi f t}{N}\right)$$

Computing complexity $O(N^2)$



Computing complexity $O(2N \log_2 N)$

https://www.momoyama-usagi.com/entry/math-seigyo14

Bit flipping

```
julia> a
3-element BitVector:
 0
 1
 0

julia> .~a
3-element BitVector:
 1
 0
 1
```

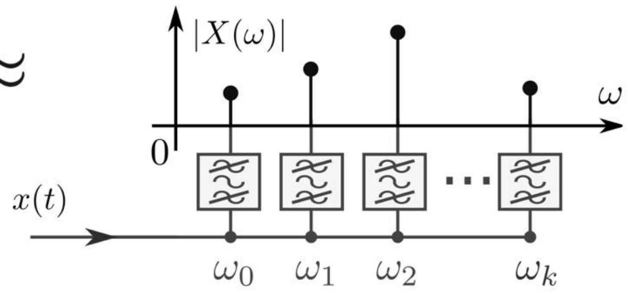
Analogy to analogue circuit

Fourier transformation:

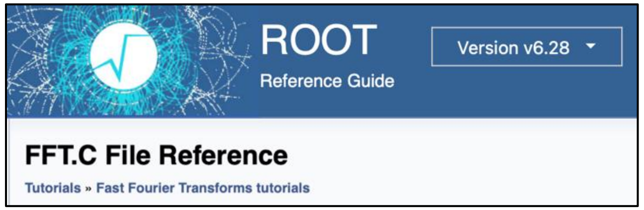
$$X(\omega_k) = \int_{t=0}^{l} x(t)\cdot e^{-j\omega t}dt$$

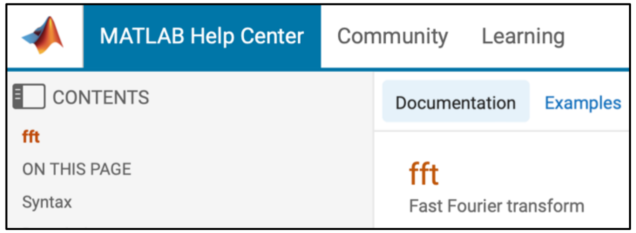
$\approx$

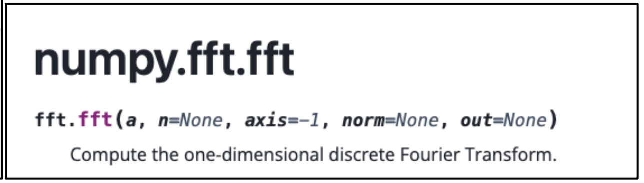
Array of bandpass filters

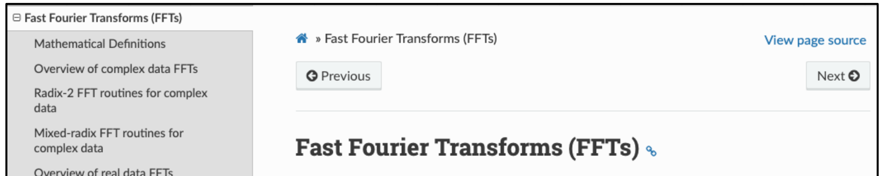


M. Betz PhD thesis









- I usually use FFT library of any programming languages
- Did I implement it when I was a student (?)

FFT → Power Spectral Density (PSD)

$$RF(t) = I(t)\cos(\omega t) + Q(t)\sin(\omega t)$$

$$\widetilde{RF}(\omega) = \int_{t=0}^{T} RF(t)e^{i\omega t}dt$$

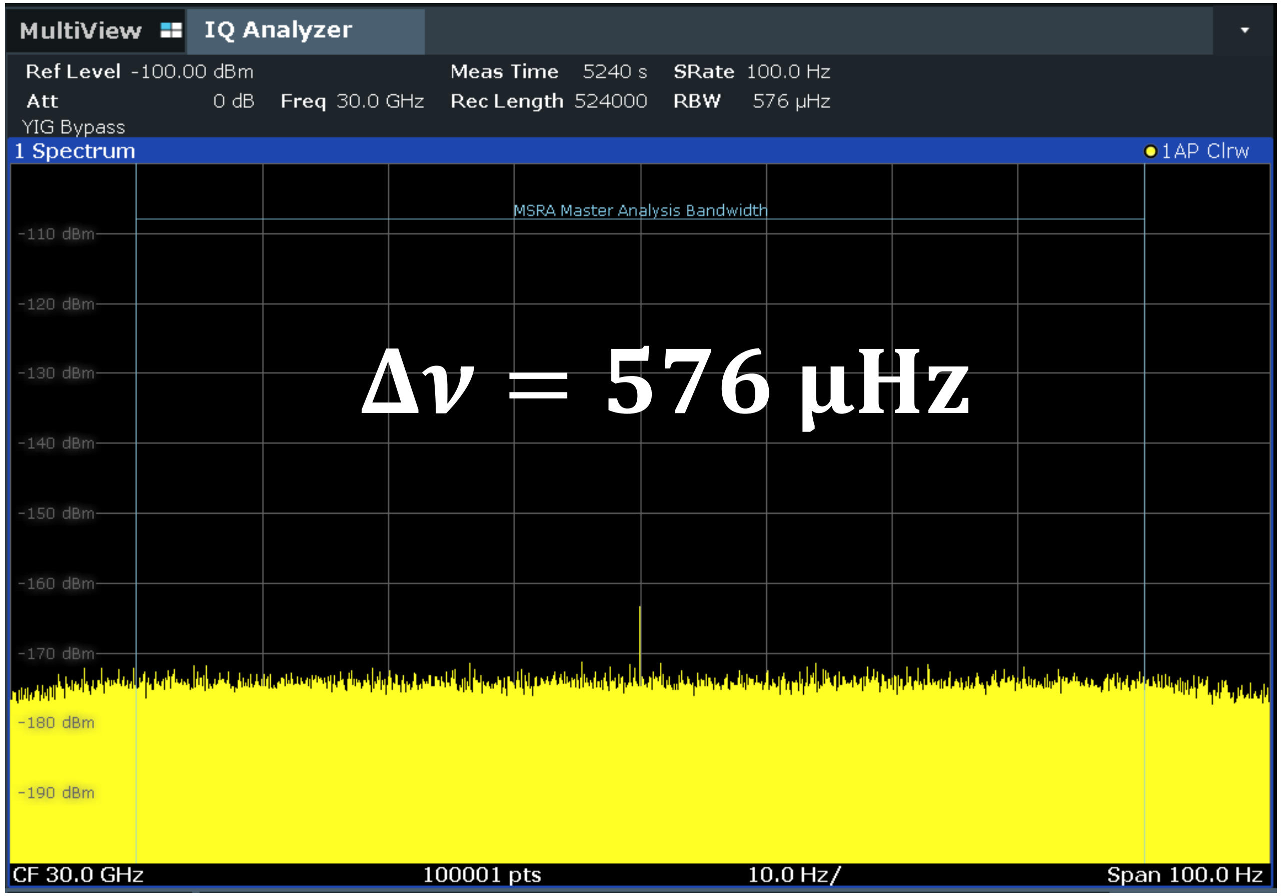
$$PSD(\omega) = \tilde{I}(\omega)^2 + \tilde{Q}(\omega)^2$$

Power is evaluated by software!

Physical unit $\mathrm{W/Hz} = \mathrm{dBm/Hz}$

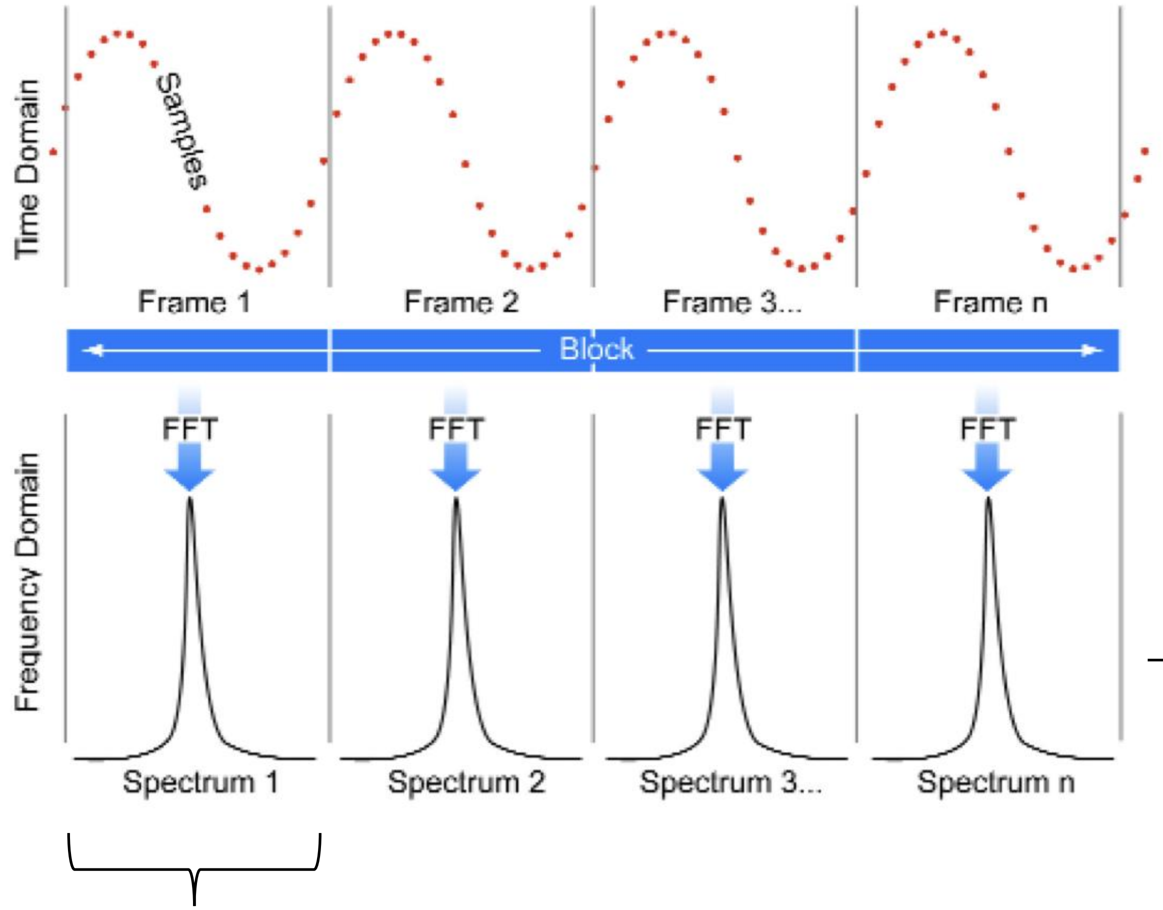
With resolution bandwidth $\Delta\nu = 1/T$

$$P_N = PSD \cdot \Delta\nu \; [\mathrm{dBm} = \mathrm{W}]$$



→ Noise power in linear system (LNA, mixer, I/Q demodulation, FFT: all linear)

Coherent and incoherent sum of spectra



FFT of time domain raw data to generate PSD

$$\widetilde{RF}(\omega) = \int_{t=0}^{T} RF(t) e^{i\omega t} dt \rightarrow PSD(\omega) = \tilde{I}(\omega)^2 + \tilde{Q}(\omega)^2$$

Average over a lot of PSDs

$$\frac{1}{N}\sum_{j=0}^{N-1} PSD_j \qquad P_N \propto \frac{1}{\sqrt{N}} \propto \frac{1}{\sqrt{T_{total}}}$$

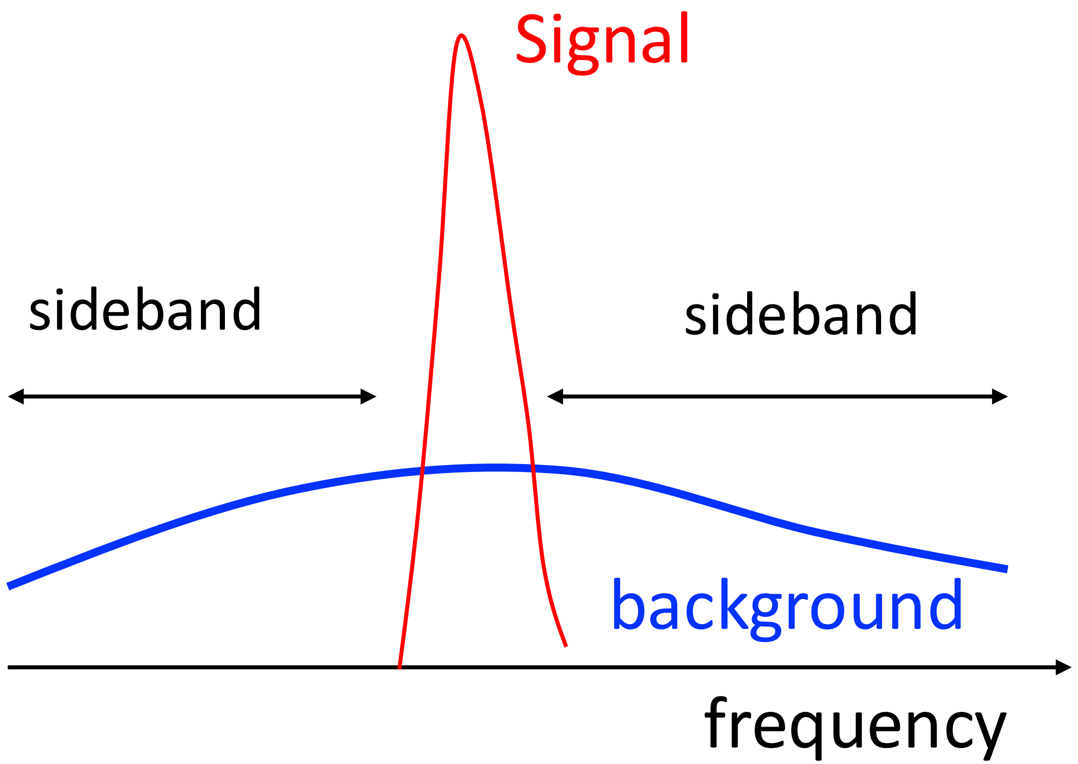
Incoherent sum

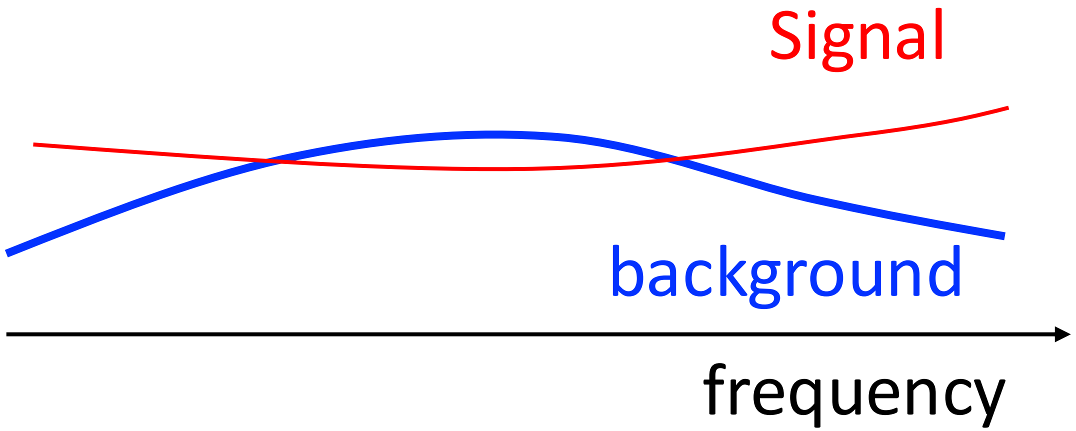
NEP

coherent sum

$$P_N = PSD(\omega)\Delta\nu \propto \frac{1}{T}$$

Narrow-band vs broad-band signal

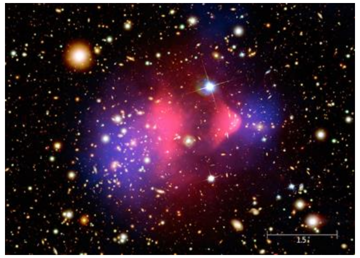


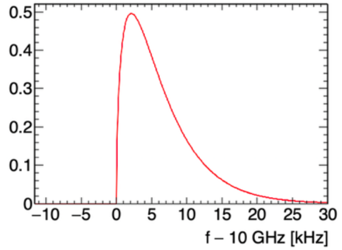


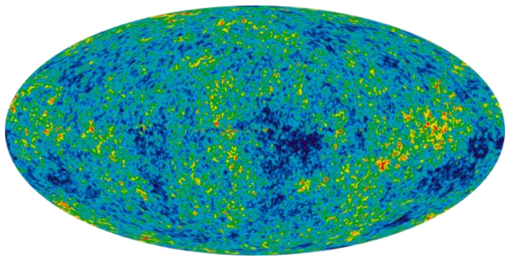
- Frequency information helps a lot to eliminate sideband in off-line analysis
- Otherwise bandpass filter in front-end to cut sidebands

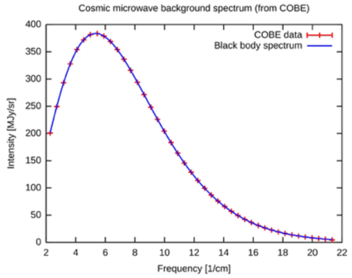
- Frequency information is useless to distinguish signal from background
- Simply sensitive sensor & low background measurement

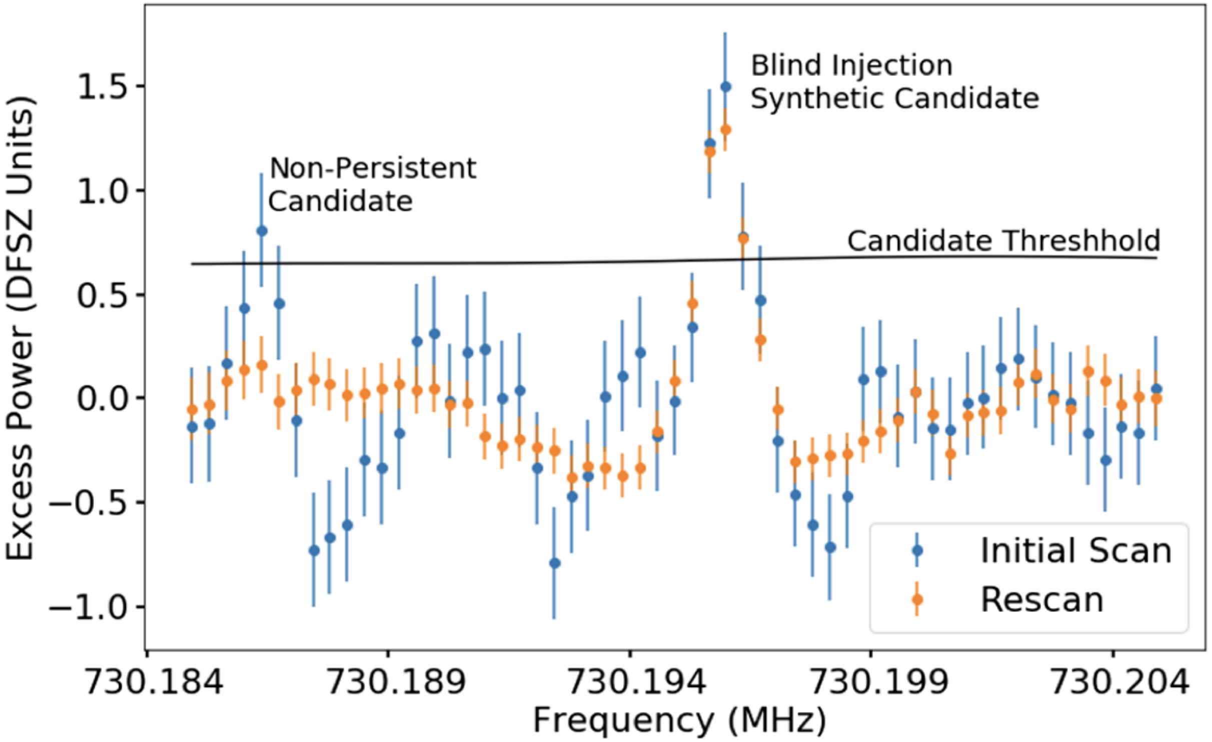
Example: narrow-band vs broad-band MW

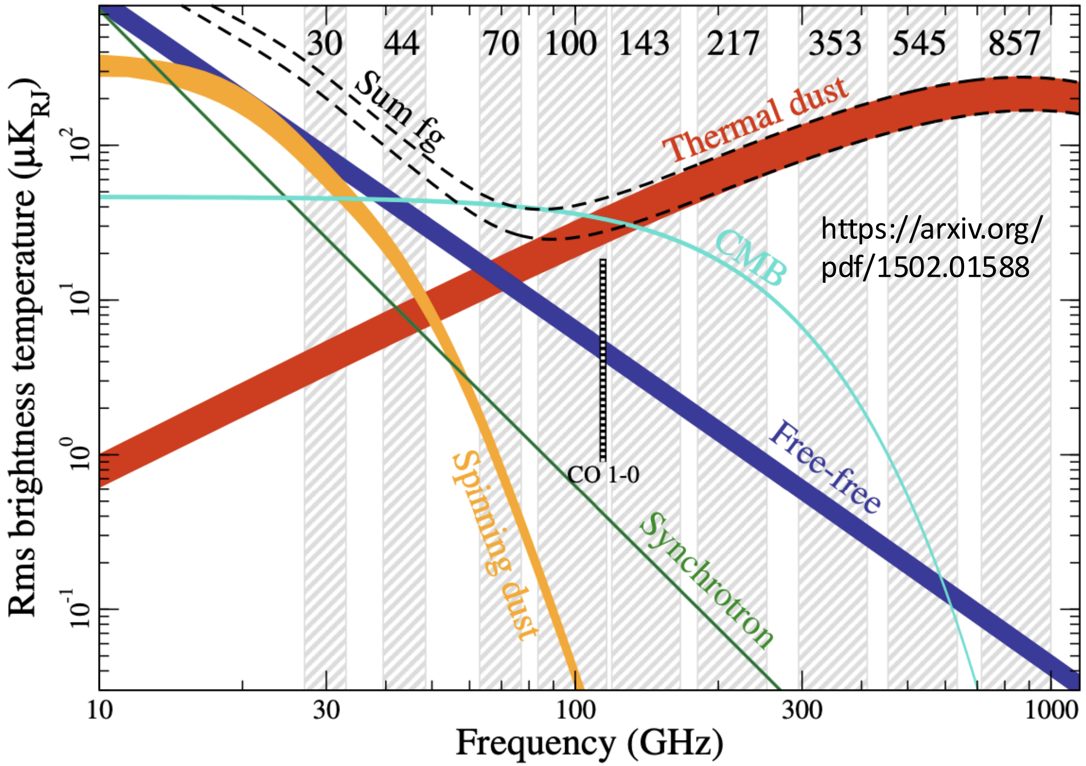








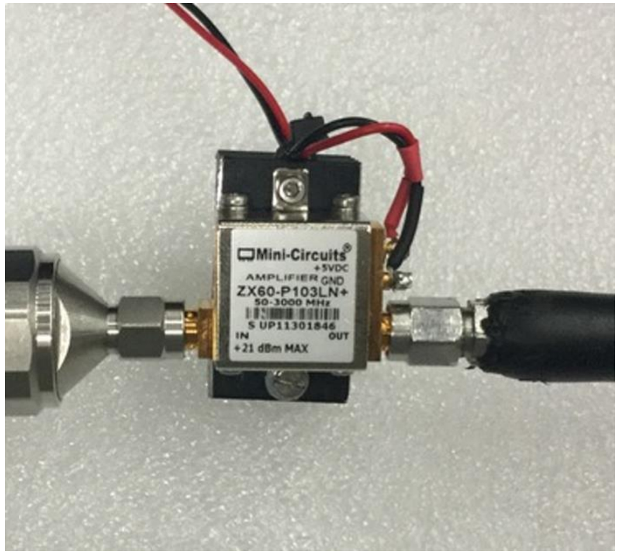


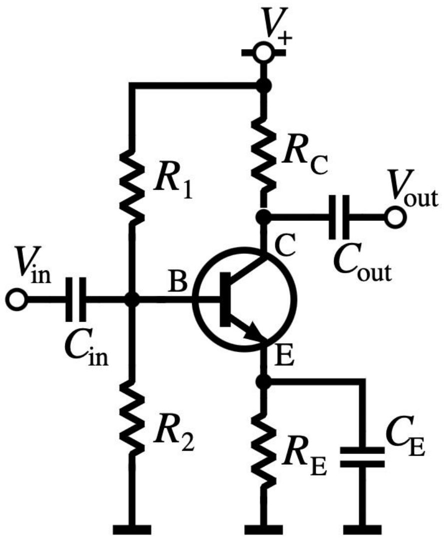


Axion experiments dramatically benefit from narrow-band detection scheme
→ Introduction of quantum sensors is nontrivial

CMB does not gain by narrow band and number of 2.7 K photons is a lot
→ Sensitive bolometer (TES, MKID, …)

Johnson Nyquist Thermal Noise

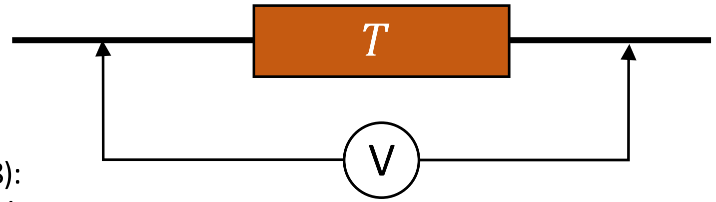


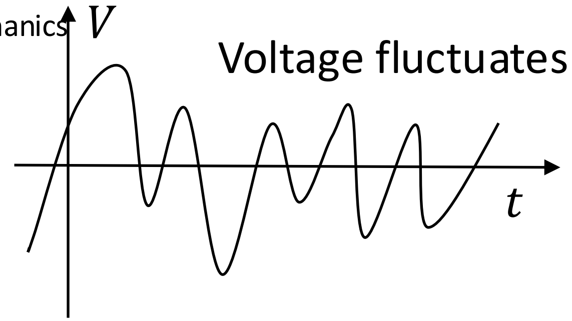


J. B. Johnson Phys Rev 32 97 (1928): Experimental discovery of the relation

H. Nyquist Phys Rev 32 110 (1928): Thermodynamics + statistical mechanics of **bosonic** modes

Any conductor at temperature T





$$P_N = \frac{\langle V^2 \rangle}{4\mathrm{R}}$$

"Blackbody radiation" of electromagnetic waves inside a 1D conductor

$$\langle V^2 \rangle \Delta\nu \sim 4R\Delta\nu \frac{h\nu}{e^{h\nu/k_B T} - 1} \xrightarrow{h\nu \ll k_B T} 4R k_B T \Delta\nu$$

Rayleigh Jeans

Noise PSD: $N = \frac{\langle V^2 \rangle}{4\mathrm{R}\Delta\nu} \sim k_B T$ [W/Hz]

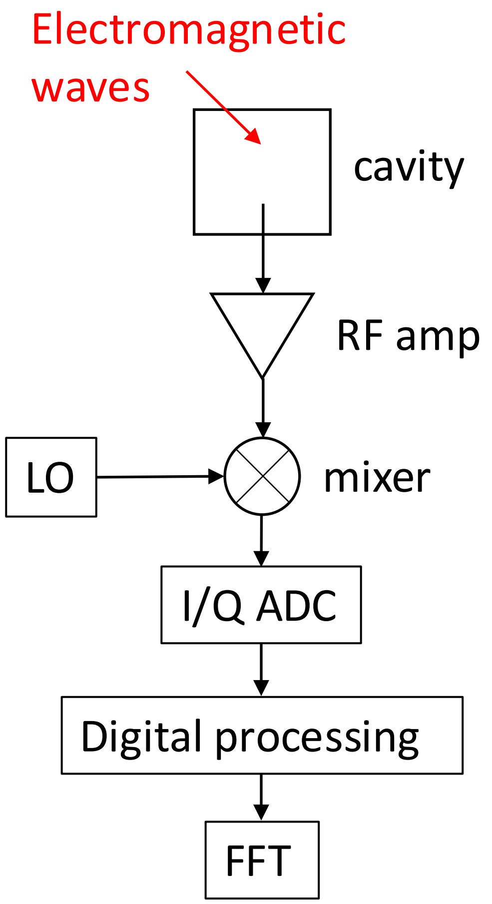
Average number of thermal photons

$$\bar{n} = \frac{1}{e^{h\nu/k_B T} - 1}$$

was derived by Planck via quantum statistics

Power is OK but what is the corresponding classical waves?

Standard Quantum Limit of linear detection scheme



$$RF(t) = I(t)\cos(\omega t) + Q(t)\sin(\omega t)$$

The analogue + digital system composed of

- Linear amplifier
- Heterodyne Mixer
- I/Q digitization
- FFT

cannot resolve power density below

$$P_{SQL} = \hbar\omega \ [\mathrm{W/Hz}]$$

For example,

- 1 GHz → $6.6 \times 10^{-25} [\mathrm{W/Hz}] = 48\ \mathrm{mK}$
- 20 GHz → $1.3 \times 10^{-23} [\mathrm{W/Hz}] = 960\ \mathrm{mK}$

Cf) 300 K blackbody radiation $4.1 \times 10^{-21} [\mathrm{W/Hz}]$

Quantum technology to overcome SQL → Part 3

Part 2: classical detection scheme

- Boundary conditions
  - Normal conductors
  - Superconductors
  - Dielectric materials: insulator and semi-conductors
- Microwave resonators
  - Waveguide and transmission line
  - Resonant cavity
  - Fabry-Pérot resonator
- Analog and digital system
  - Amplifier, circulator, mixer & analogue down-conversion
  - I/Q sampling & digital down-conversion
- Data processing and noise
  - FFT: coherent and incoherent integral
  - Narrowband and broadband
  - Thermal noise and standard quantum limit
- Conclusion of part 2

Conclusion of Part 2

- Boundary conditions
  - Normal conductors has skin depth of µm and surface resistance of $m\Omega$
  - Superconductors has penetration depth of 100 nm and surface resistance of 10 $n\Omega$ but external magnetic fields degrade the resistance → parallel B-field is the key
  - Dielectric materials can be another material to confine microwaves → excellent performance inside magnetic field
- Microwave resonators
  - Microwaves are transmitted via Gaussian beam in free space while confined in transmission lines
  - Resonant cavities can enhance the field strength so that the conversion ratio from DM axion can also be enhanced
  - TM_{010} is the most commonly used resonant mode for DM search (as well as particle accelerators)
  - Fabry-Pérot resonators confine Gaussian beam (TEM_{00n}) and do not couple to DM axions without dielectric disks while it is useful for the relativistic axions to couple in LSW-type experiments
- Analog and digital system
  - Amplifier, circulator, mixer & analogue down-conversion & I/Q sampling & digital down-conversion gives the present state-of-the-art of microwave engineering
  - The processed data is linear to the input amplitude toward the very end of the processing
- Data processing and noise
  - Narrow-band signal can significantly gain S/N via FFT of the I/Q data (still linear !) and can linearly supress the broadband noise proportional to the integral time (coherent sum)
  - Once the resolution bandwidth reaches the signal band-with, FFT over longer integration time does not improved the S/N but simply resolve the signal into smaller binds → simple averaging of PSD improves noised power proportional to the square root of the integration time
  - The linear detection methods would eventually reach SQL

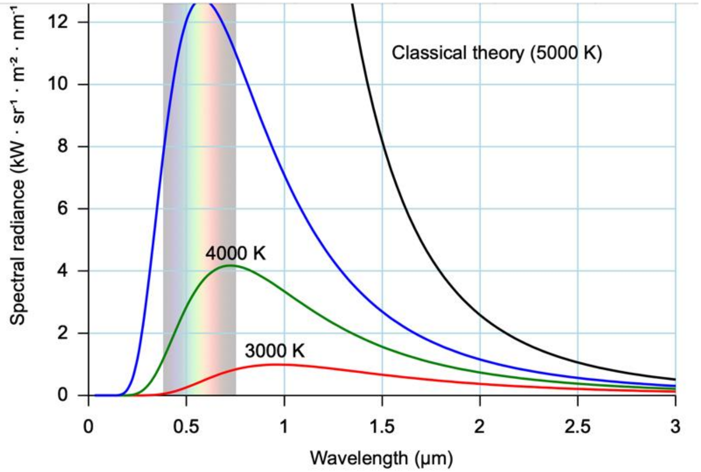
backup

dBm: conventionally used as a unit of absolute power

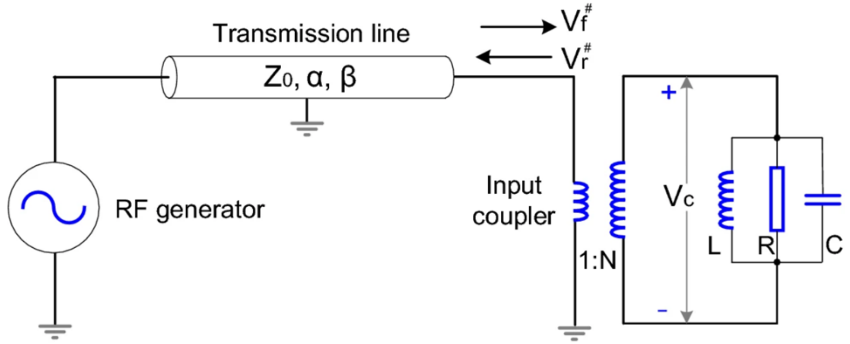
I became familiar with the concept of dB? with dBm without $10\log_{10} A/B$

Typical numbers to remember

- $120\ \mathrm{dBm} = 1\ \mathrm{GW}$: one nuclear power plant
- $110\ \mathrm{dBm} = 100\ \mathrm{MW}$: Synchrotron radiation of FCCee
- $90\ \mathrm{dBm} = 1\ \mathrm{MW}$: high power RF amplifier / oscillator
- $10\ \mathrm{dBm} = 10\ \mathrm{mW}$: "high-power" in low-level electronics
- $0\ \mathrm{dBm} = 1\ \mathrm{mW}$: reference power
- $-40\ \mathrm{dBm}$: broad-band power sensors noise floor
- $-120\ \mathrm{dBm/Hz}$: noise floor of ADC
- $-174\ \mathrm{dBm/Hz}$: black body radiation noise from 300 K
- $-212\ \mathrm{dBm/Hz}$: standard quantum limit noise of 1GHz

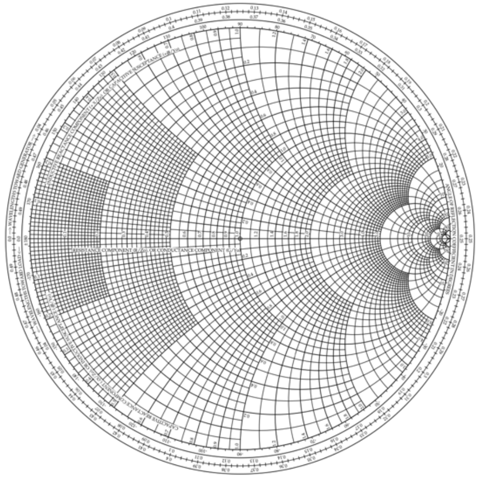


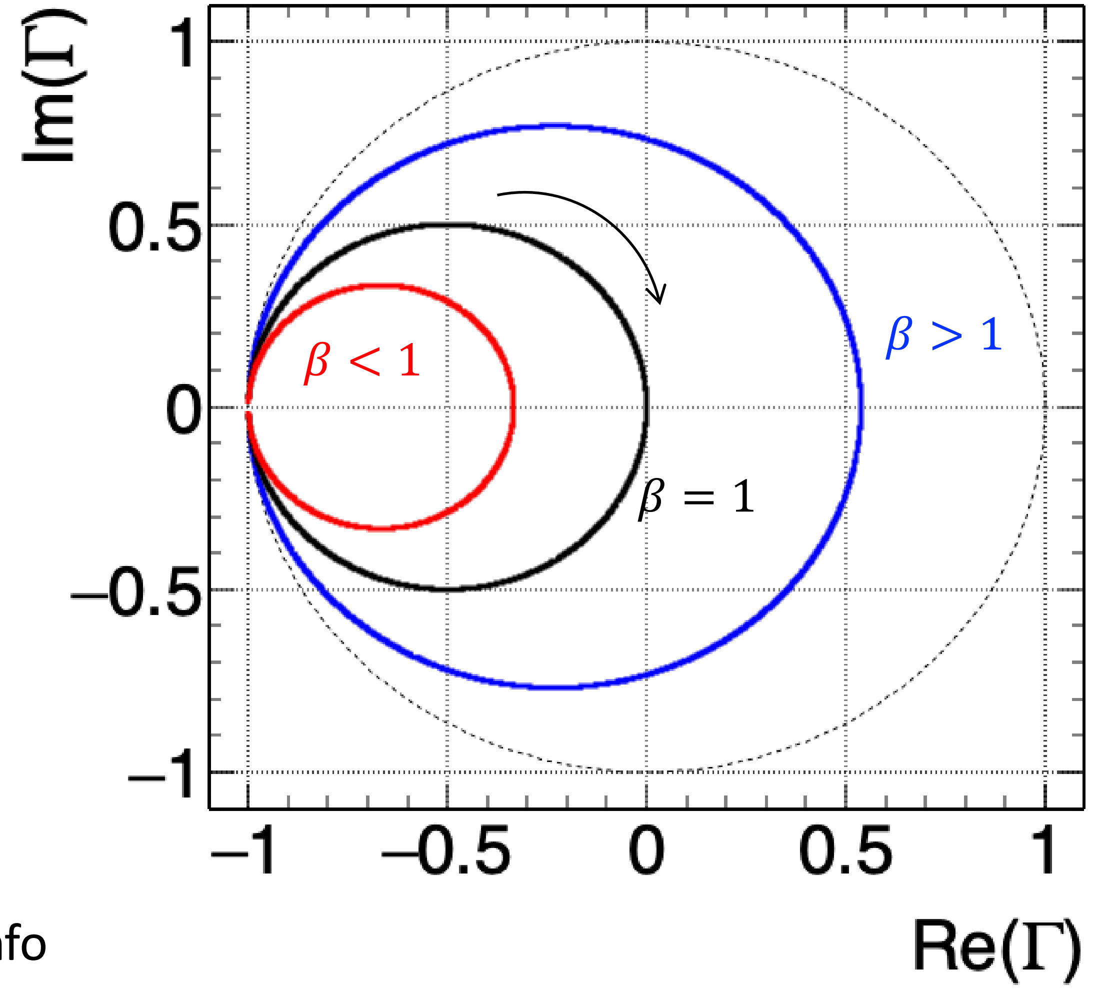
Smith chart to see the coupling



$$\Gamma = |\Gamma|e^{i\theta} = \frac{\left(1-\frac{1}{\beta}\right) - iQ_{ext}\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)}{\left(1+\frac{1}{\beta}\right) + iQ_{ext}\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)} = \frac{\beta - 1 - iQ_0\delta_\omega}{\beta + 1 + iQ_0\delta_\omega}$$

- This is a textbook example to find the coupling condition of the cavity + coupler system
- Circle smaller than diameter 1 indicates under coupling





This method is useful under two conditions:

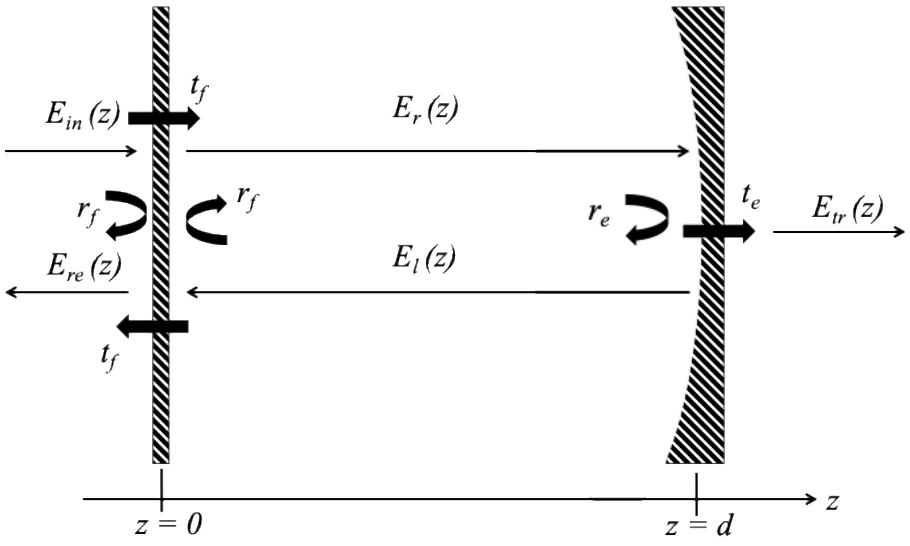
- The resonance is not super narrow
- Q_{ext} and Q_0 are not very different

In case of superconducting cavities,

$$Q_0 \sim 10^{10}, Q_{ext} \sim 10^5$$

Is a typical condition → just a large circle without not much info

Interferometer model 1/2



Transmitted and reflected power

$$P_{tr} = |E_{tr}(z)|^2 = \frac{T_f T_e}{\left(1-\sqrt{R_f R_e}\right)^2} \frac{1}{1 + F\sin^2 kd} P_{in}$$

$$P_{re} = |E_{re}(z)|^2 = \frac{\left[\frac{\sqrt{R_f} - (T_f + R_f)\sqrt{R_e}}{1-\sqrt{R_f R_e}}\right]^2 + (T_f + R_f)F\sin^2 kd}{1 + F\sin^2 kd} P_{in}$$

with

$$F \equiv \frac{4\sqrt{R_f R_e}}{\left(1-\sqrt{R_f R_e}\right)^2}$$

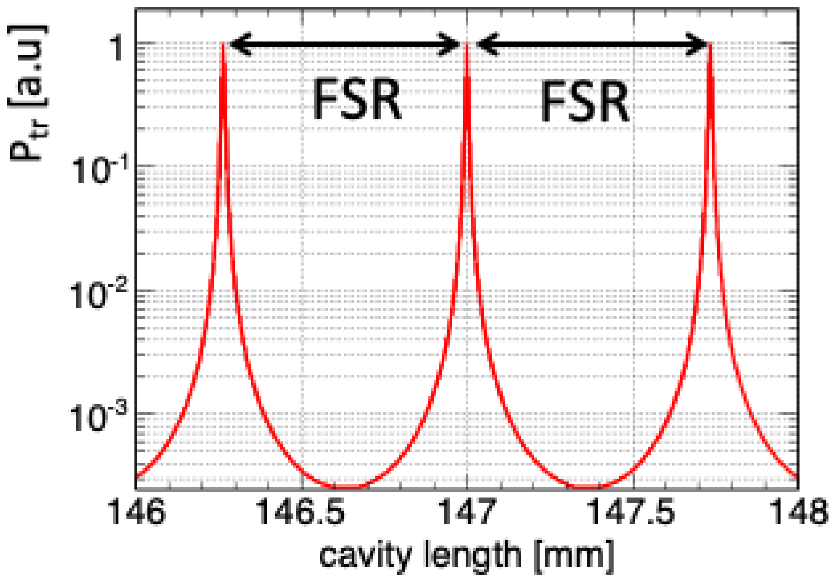
and finesse of the Fabry Pérot resonator is defined as

$$\mathcal{F} = \frac{\pi}{2}\sqrt{F} \sim \frac{2\pi}{1 - R_f R_e}$$

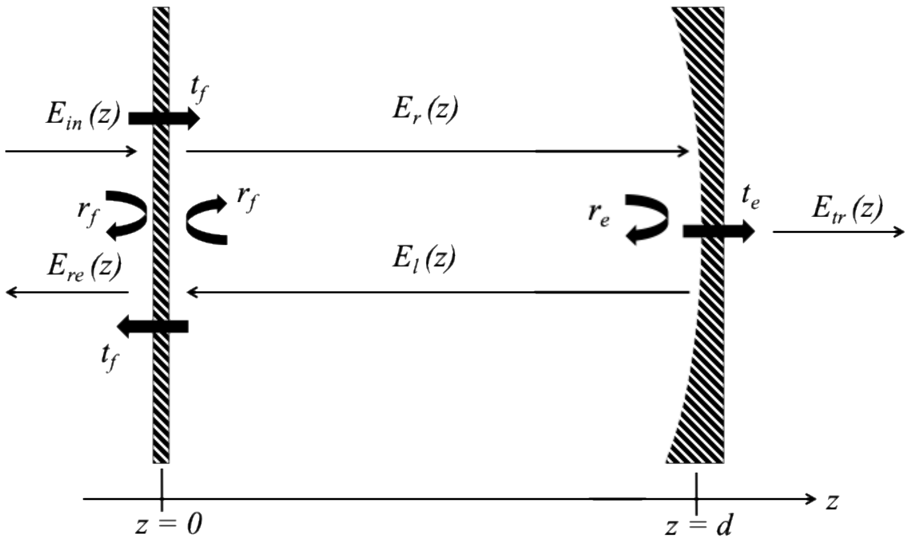
Finesse is an experimental observable

$$\mathcal{F} = \frac{\pi}{\mathrm{FWHM\ [radian]}} = \frac{\mathrm{FSR}[mm]}{\mathrm{FWHM}\ [mm]}$$

Free Spectral Range



Interferometer model 2/2

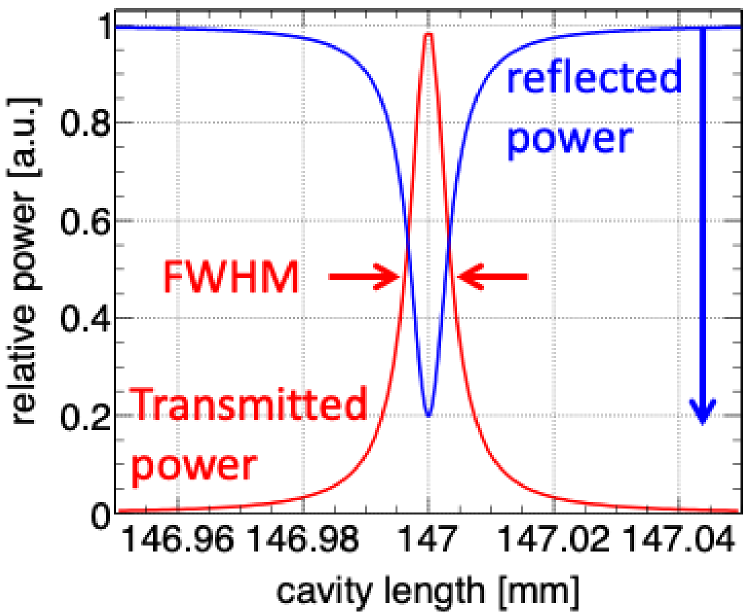


Coupling

$$P_{re}(\omega_0) = \left[\frac{\sqrt{R_f} - (T_f + R_f)\sqrt{R_e}}{1 - \sqrt{R_f R_e}}\right]^2$$ ***On resonance***

$$P_{re}(\omega \gg \omega_0) = \left[\frac{\sqrt{R_f} + (T_f + R_f)\sqrt{R_e}}{1 + \sqrt{R_f R_e}}\right]^2$$ ***Off resonance***

Reflection coefficient $|\Gamma| = |S_{11}| = \sqrt{P_{re}(\omega_0)}/\sqrt{P_{re}(\omega \gg \omega_0)}$



Coupling coefficient

$$\beta = \frac{1 \mp |\Gamma|}{1 \pm |\Gamma|}$$

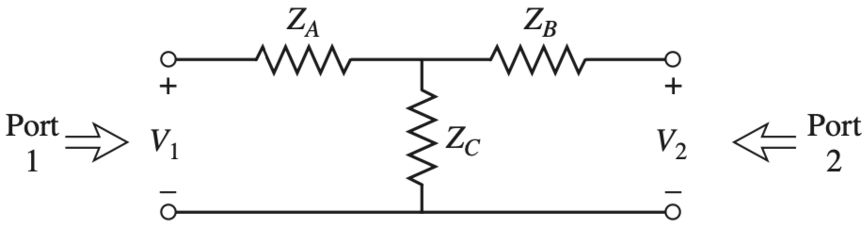
Upper sign for under coupling,
lower sign for over coupling

Standing wave's power…*circulating power* (?) stored energy (??)

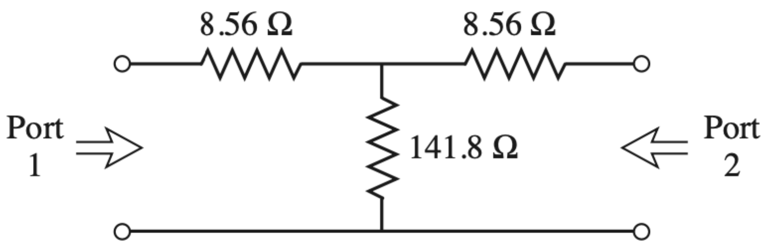
$$P_{std}(?) = |E_{std}(z)|^2 = T_f \frac{\left(\frac{1 + \sqrt{R_e}}{1 - \sqrt{R_f R_e}}\right)^2 - \frac{1}{\sqrt{R_f}} F \sin^2 k(z - d)}{1 + F \sin^2 kd}$$

Attenuators

- Control power level
- Reduce noise level from outside
- Attenuation and tolerable power to specify
- Hopefully axions do not burn it unlike accelerators ☺ (calculate power!)



Ex) 3 dB (half power) attenuator for 50 Ω line

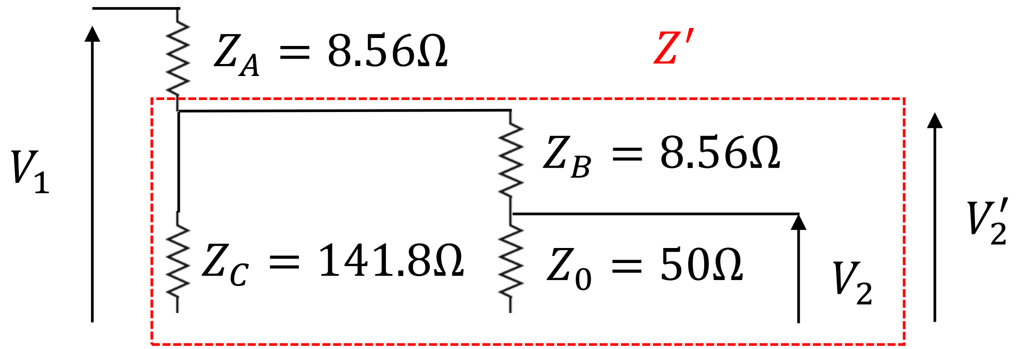


Useful intermediate impedance: $Z' \equiv \dfrac{Z_C(Z_B + Z_0)}{Z_C + (Z_B + Z_0)} = 41.44\Omega$

Suppose no additional reflection to port 1 and 2

$$S_{11} = \frac{V_1^-}{V_1^+} = \frac{Z_{in}^{(1)} - Z_0}{Z_{in}^{(1)} + Z_0} \qquad Z_{in}^{(1)} = Z_A + Z' = 50\Omega \rightarrow S_{11} = 0$$

Equivalent circuit



$$S_{21} = \frac{V_2^-}{V_1^+} = \frac{V_2}{V_1}$$

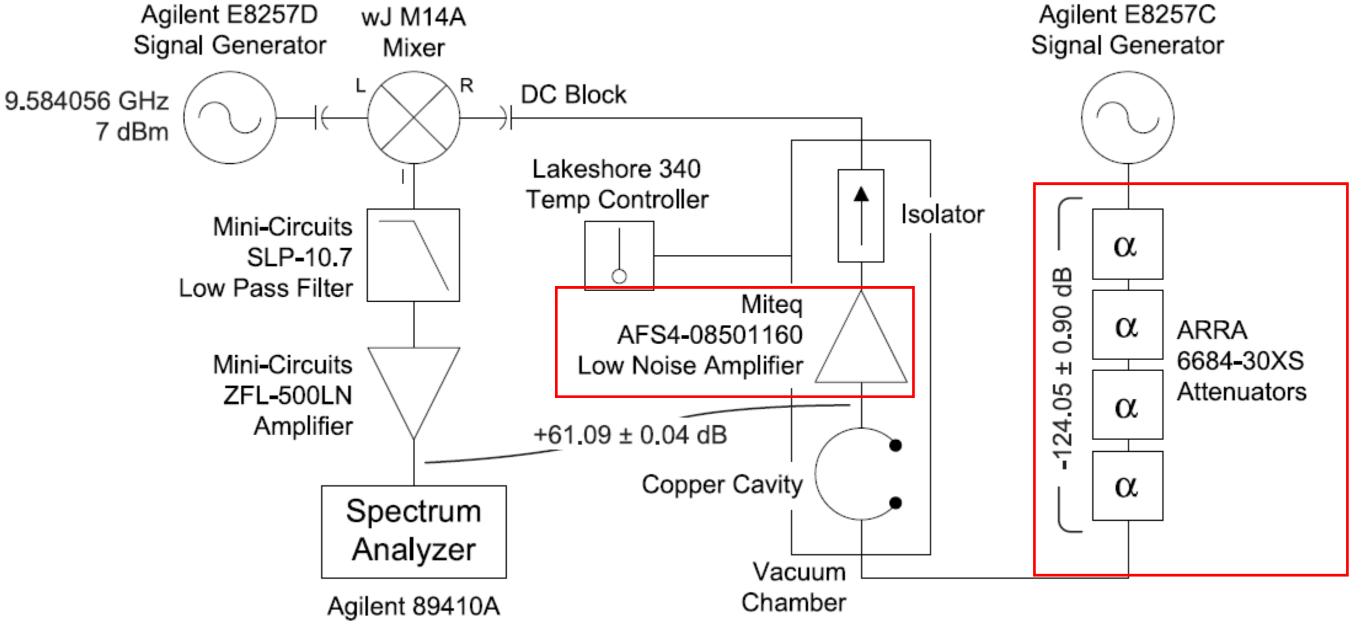
$$\left.\begin{aligned} V_2 &= \frac{Z_0}{Z_B + Z_0} V_2' \\ V_2' &= \frac{Z'}{Z_A + Z'} V_1 \end{aligned}\right\}$$

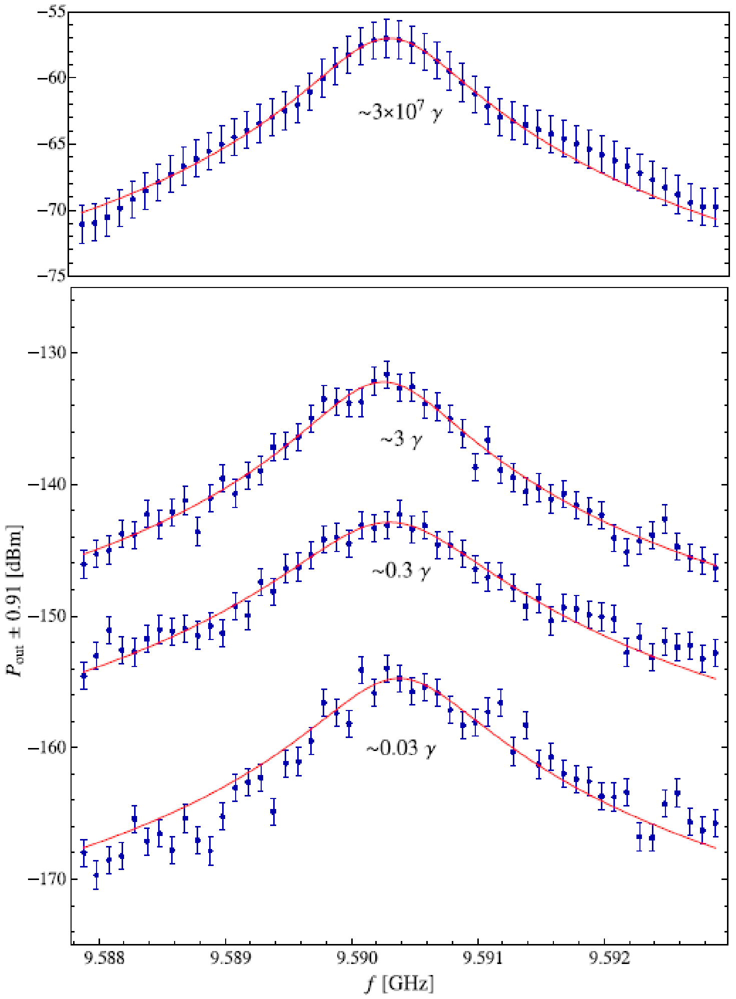
$S_{21} = 0.71$

$|S_{21}|^2 = 0.50$

Cf) attenuator does not make $|1\rangle$ photon state

J. G. Hartnett et al Phys Lett B 698 346 (2011)

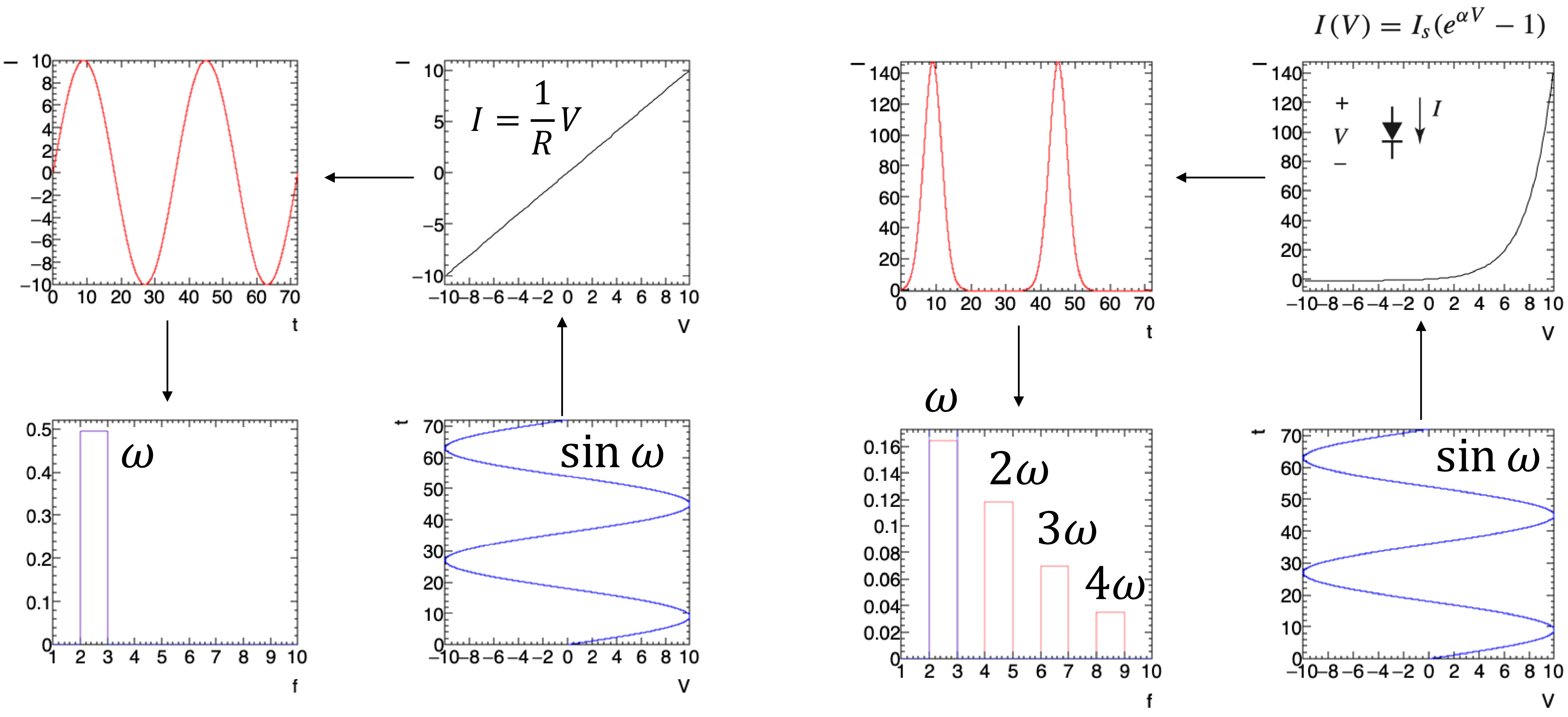




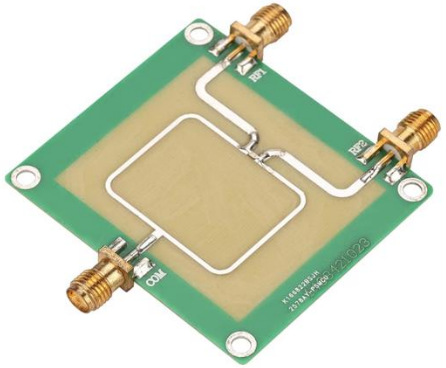
P_{in} [dBm]	$\#\gamma$ in cavity	Q_L
−55	$\sim 3 \times 10^7$	8800
−125	~ 3	8900
−135	~ 0.3	7100
−145	~ 0.03	8200

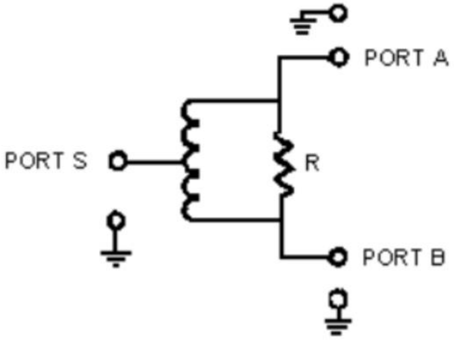
- Attenuator does not change statistical nature (coherent state → Poissonian) but simply reduce mean number of photons ($= |E|^2$) so does not go into true quantum regime
- True single photon state $|1\rangle$ may have distinct behavior in interference

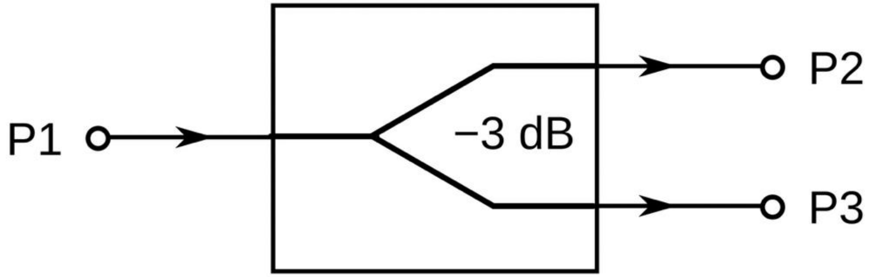
Demo: nonlinearity in diode → harmonics generation

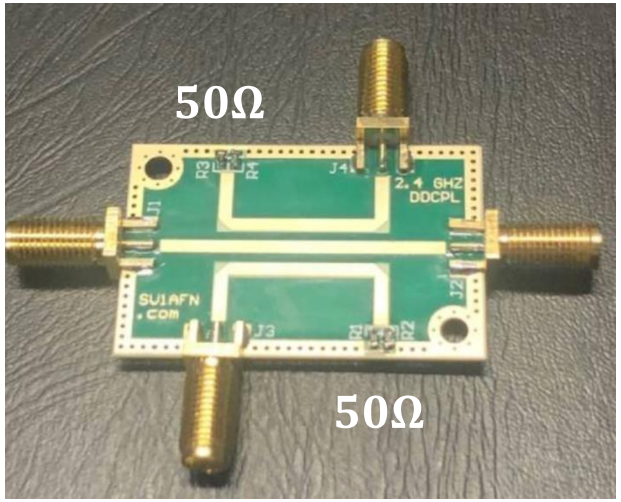


Splitter / directional coupler

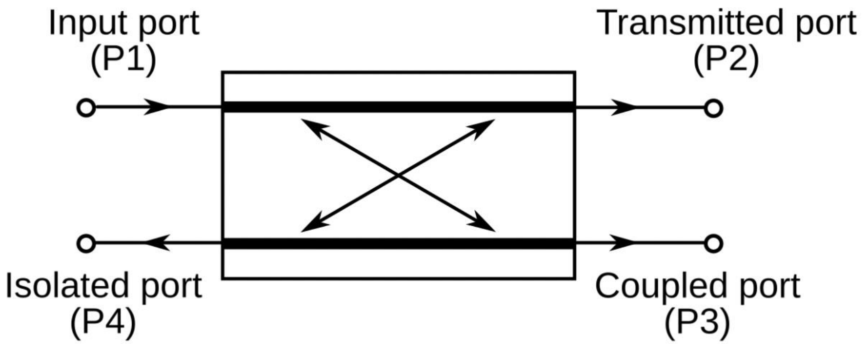


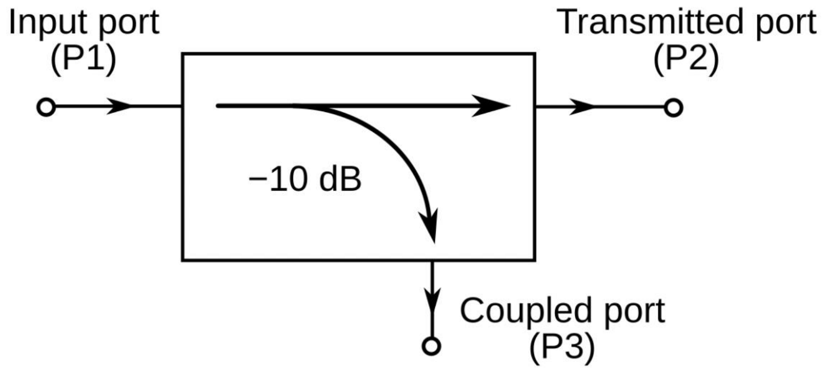






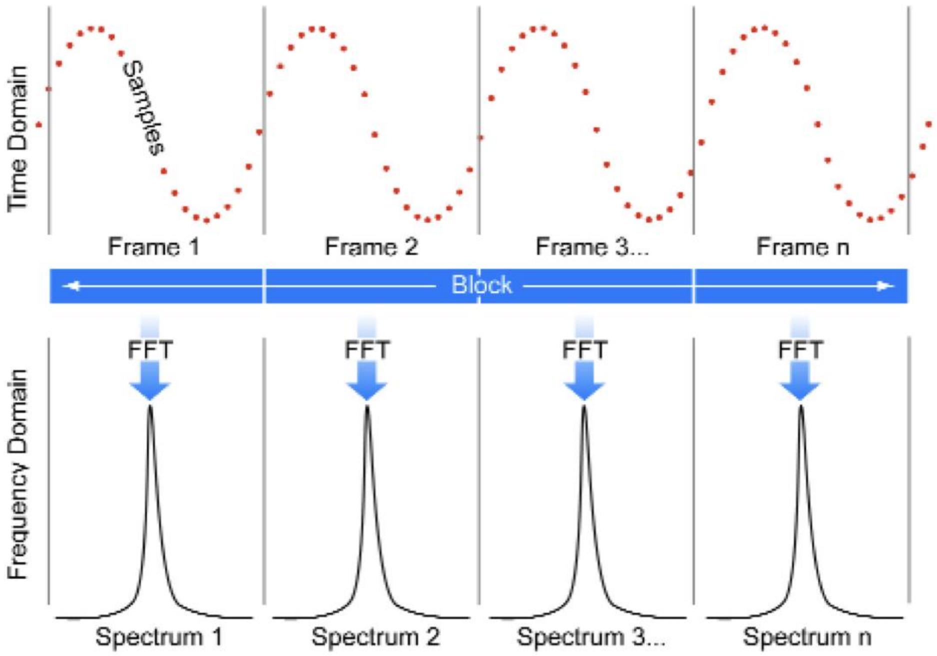
Coupling P_3/P_1
Isolation P_4/P_1
Directivity P_4/P_3
Insertion loss P_2/P_1
(in dB)

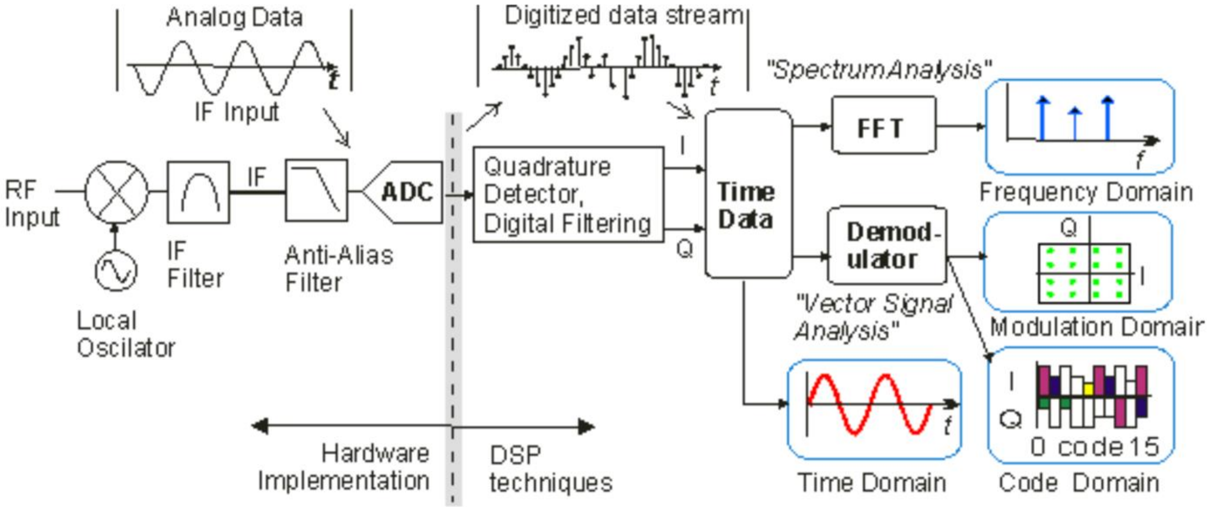


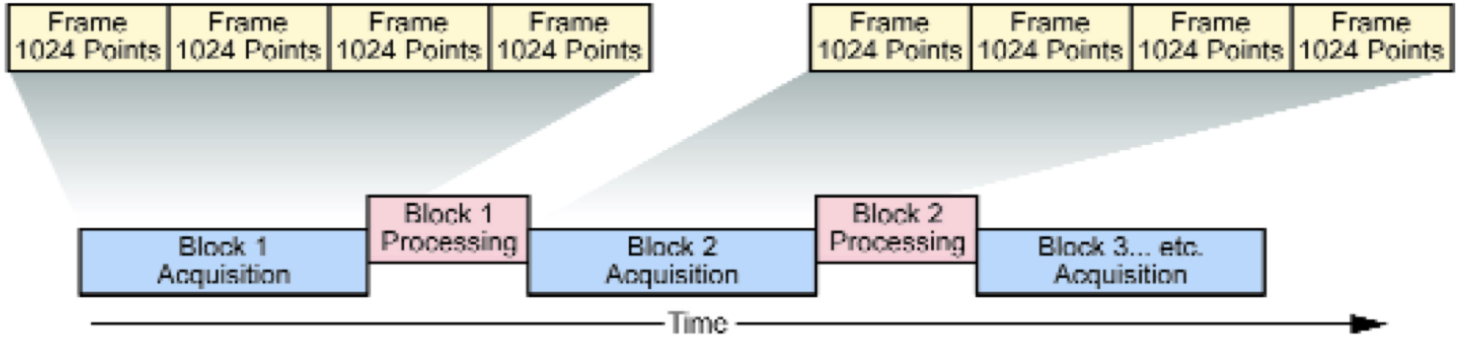


- Tried to reverse engineer them via CST MW Studio → failed
- Please terminate non-used ports ☺

Real-time Spectrum Analyzer ~Vector Signal Analyzer)



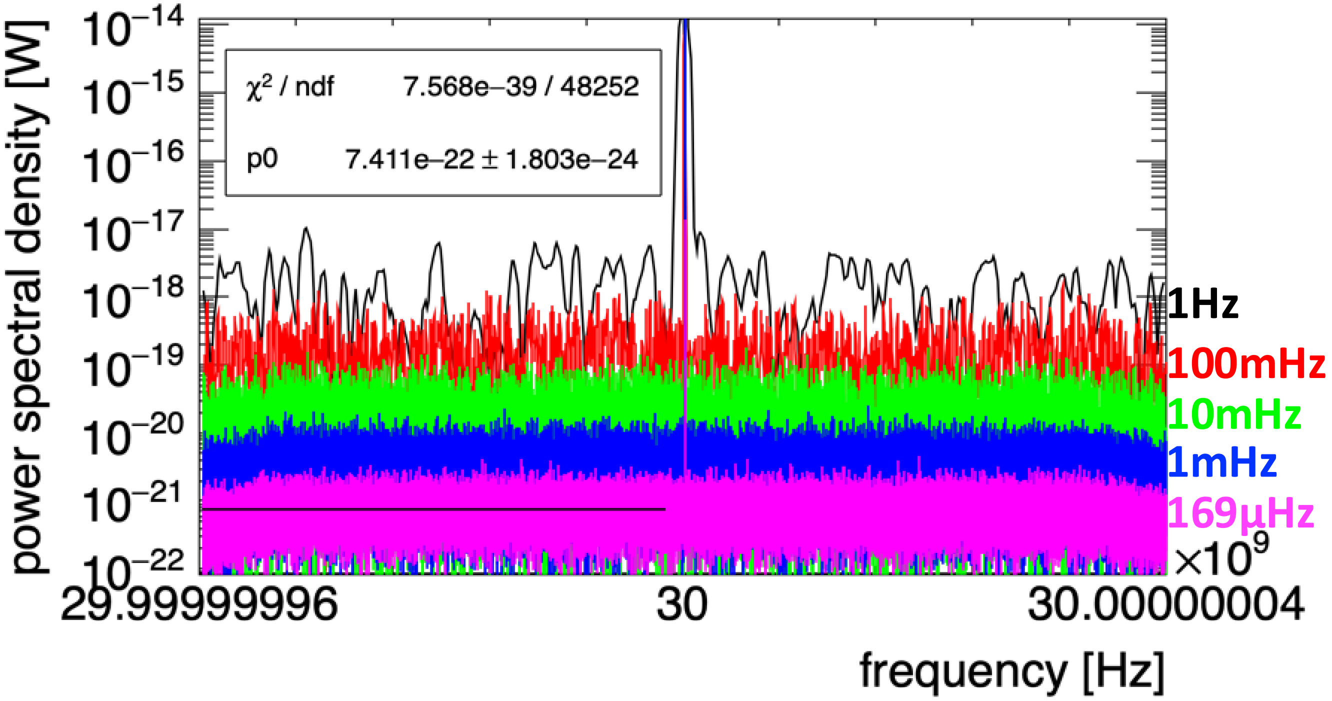


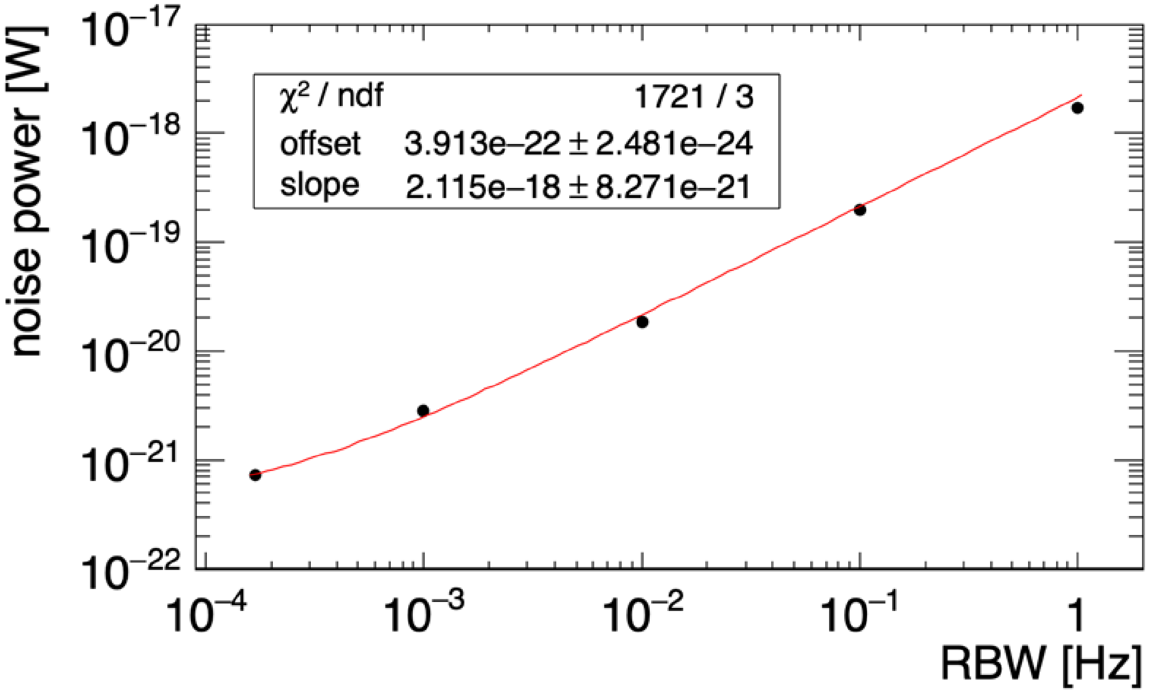


- Real-time SA stores acquired data in a ring buffer memory and perform FFT
- From time to time, it reads out the buffer and saves it in a more secure place (HD? RAM?) and re-acquire data in the buffer
- The data processing time causes dead time of data acquisition

https://svelto.faculty.polimi.it/didattica/materiale_didattico/materiale%20didattico_MRF/appnote/real_time_spectrum_analysis.pdf
https://helpfiles.keysight.com/csg/89600B/Webhelp/Subsystems/concepts/content/concepts_types_spec_an.htm#:~:text=A%20real%2Dtime%20spectrum%20analyzer,not%20in%20"real%2Dtime.

Super narrow band detection is possible





- If the buffer is long enough, FFT time $T \gg 1\ s$ is feasible → $\Delta\nu = 1/T \ll 1\ s$
- By using R&S FSW43, we demonstrated $\Delta\nu \sim 169\ \mu\mathrm{Hz}$ (1.6 hours) at 30 GHz
- One day FFT is theoretically possible
  - Demonstrated by CROWS (1-3 GHz) to reach $\Delta\nu \sim 1\ \mu\mathrm{Hz}$
- Then the questions is phase stability over such a long period
- AM et al ANNALEN DER PHYSIK 2023, 536, 2200619