PHASE TRANSITIONS OF A NONCOMMUTATIVE REISSNER-NORDSTRÖM BLACK HOLE

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Gravitational Waves and the Early Universe: Accelerated Expansion, Dynamical Inhomogeneity, and Beyond;
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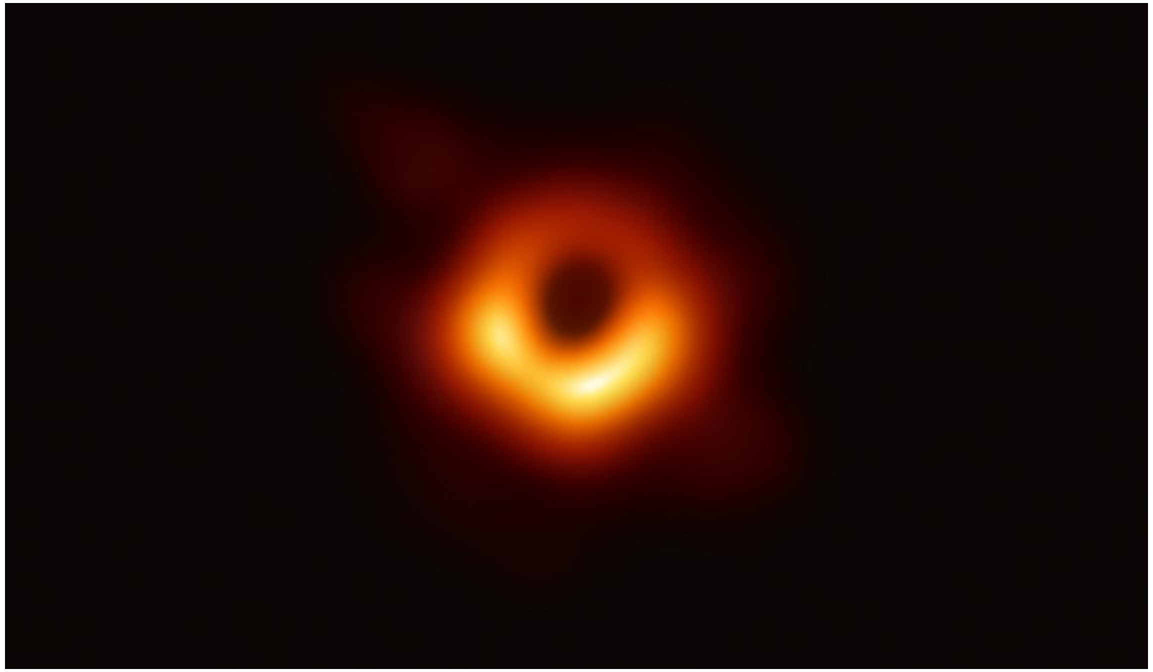
Introduction

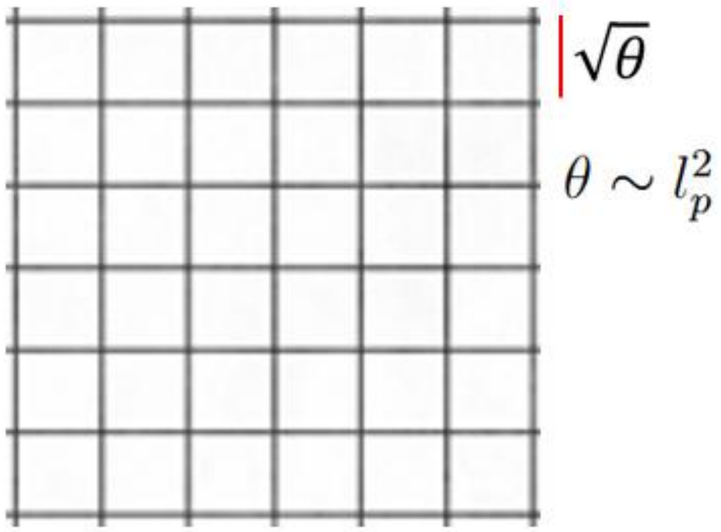


BHs represent thermodynamic systems capable of undergoing phase transitions [1].
We consider a geometric description of phase transitions [2].

Reissner-Nordström noncommutative metric

A noncommutative (NC) spacetime satisfies $[\hat{x}^\mu, \hat{x}^\nu] \neq 0$ ...(I)

No punctual structures!



$\rho(r) = M\delta^3(r)$ is no longer valid

new distributions [3,4]:

$$\rho_\theta(r) = \frac{M}{(4\pi\theta)^{3/2}} \exp\left(-\frac{r^2}{4\theta}\right)$$ ...(II)

✓ $$\rho^{(\Theta)} = \frac{M\sqrt{\Theta}}{\sqrt{\pi^3}(r^2+\pi\Theta)^2}$$ ...(III) within black hole thermodynamics it is simple to use a Lorentzian density (III)

From the field equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi\left(T^{(em)}_{\mu\nu} + T_{\mu\nu}{}^{(\Theta)}\right)$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right) = \rho_e(r,\Theta)\delta^\nu_0$$

…(IV)

with $\rho_e(r,\Theta) = \frac{4Q\sqrt{\Theta}}{\sqrt{\pi}(r^2+\pi\Theta)^2}$, we find the electric field

$$E(r,\Theta) = \frac{2Q}{\sqrt{\pi}r^2}\left[\frac{1}{\sqrt{\pi}}\arctan\left(\frac{r}{\sqrt{\pi\Theta}}\right) - \frac{r\sqrt{\Theta}}{\pi\Theta + r^2}\right]$$

…(V)

Also, we write the NC energy-momentum tensor as an anisotropic fluid [3]

$$T^{\mu}_{\ \nu}{}^{(\Theta)} = diag\left(-\rho^{(\Theta)}, \rho^{(\Theta)}, p^{(\Theta)}_\theta, p^{(\Theta)}_\phi\right)$$

…(VI)

with the pressures satisfying $p^{(\Theta)}_\theta = p^{(\Theta)}_\phi$ and $p^{(\Theta)}_\theta = -\rho^{(\Theta)} + \frac{r}{2}\partial_r\rho^{(\Theta)}$

Under the assumption of the energy density of the electric charge $\rho_Q^{(\Theta)} = \frac{E^2}{8\pi}$,
using (V) together with the spherically symmetric metric

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

in the Einstein field eqs. (IV), we obtain the NCRN metric

$$f(r) = 1 - \frac{c}{r} + \frac{4\pi M\sqrt{\pi\Theta} - 2Q^2}{\pi^2(\pi\Theta + r^2)} - \frac{2}{\pi^2 r}\left(2\pi M + \frac{Q^2}{\sqrt{\pi\Theta}}\right)\arctan\left(\frac{r}{\sqrt{\pi\Theta}}\right) + \frac{4Q^2}{\pi^2 r^2}\arctan^2\left(\frac{r}{\sqrt{\pi\Theta}}\right)$$

which can be approximated for $r \gg 1$ as [4]

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{8\sqrt{\Theta}M}{\sqrt{\pi}r^2} - \frac{4\sqrt{\Theta}Q^2}{\sqrt{\pi}r^3} \qquad \ldots\text{(VII)}$$

Geometrothermodynamics (GTD)

Among the geometric descriptions of thermodynamics, GTD gives a Legendre invariant formalism. Thus, we consider the following metrics for quasihomogenous systems [5]:

$$g^{\mathrm{I}} = \sum_{a,b,c=1}^{n} \left(\beta_c E^c \frac{\partial \Phi}{\partial E^c}\right) \frac{\partial^2 \Phi}{\partial E^a \partial E^b} \mathrm{d}E^a \mathrm{d}E^b,$$

$$g^{\mathrm{II}} = \sum_{a,b,c,d=1}^{n} \left(\beta_c E^c \frac{\partial \Phi}{\partial E^c}\right) \eta_a^d \frac{\partial^2 \Phi}{\partial E^b \partial E^d} \mathrm{d}E^a \mathrm{d}E^b,$$

$$g^{\mathrm{III}} = \sum_{a,b=1}^{n} \left(\beta_a E^a \frac{\partial \Phi}{\partial E^a}\right) \frac{\partial^2 \Phi}{\partial E^a \partial E^b} \mathrm{d}E^a \mathrm{d}E^b.$$

…(VIII)

defined for the equilibrium space $\mathcal{E}$, which is a subspace of the phase space $\mathcal{T}$ such that

$$\varphi : \mathcal{E} \to \mathcal{T}, \quad \text{i.e.,} \quad Z^A = \{\Phi(E^a), E^a, I_a(E^b)\}$$

GTD dictionary

Curvature ⇒ Thermodynamic interaction

Geodesics ⇒ Quasistatic processes

Divergence of the Ricci scalar ⇒ Phase transitions

Black hole thermodynamics

From the NCRN metric (VII), we can write the mass

$$M = \frac{\pi Q^2 \sqrt{S} + \sqrt{S^3} - 4\pi Q^2 \sqrt{\Theta}}{2\sqrt{\pi S}(\sqrt{S} - 4\sqrt{\Theta})} \qquad \text{...(IX)}$$

Also, the NCRN metric satisfies the relation

$$M(\lambda^{\beta_S} S, \lambda^{\beta_Q} Q, \lambda^{\beta_\Theta} \Theta) = \lambda^{\beta_M} M(S, Q, \Theta)$$

Here, $\beta_S = \beta_\Theta, \ \beta_S = 2\beta_Q, \ \beta_S = 2\beta_M$ ...(X)

Phase transitions

The phase transition conditions are derived from (VIII) for a 3D case *M=M(S,Q,Θ)*

$$I:\frac{\partial^2 M}{\partial Q^2}\left(\frac{\partial^2 M}{\partial S\partial\Theta}\right)^2-2\frac{\partial^2 M}{\partial Q\partial\Theta}\frac{\partial^2 M}{\partial S\partial\Theta}\frac{\partial^2 M}{\partial S\partial Q}+\frac{\partial^2 M}{\partial\Theta^2}\left(\frac{\partial^2 M}{\partial S\partial Q}\right)^2+\frac{\partial^2 M}{\partial S^2}\left(\left(\frac{\partial^2 M}{\partial Q\partial\Theta}\right)^2-\frac{\partial^2 M}{\partial Q^2}\frac{\partial^2 M}{\partial\Theta^2}\right)=0$$

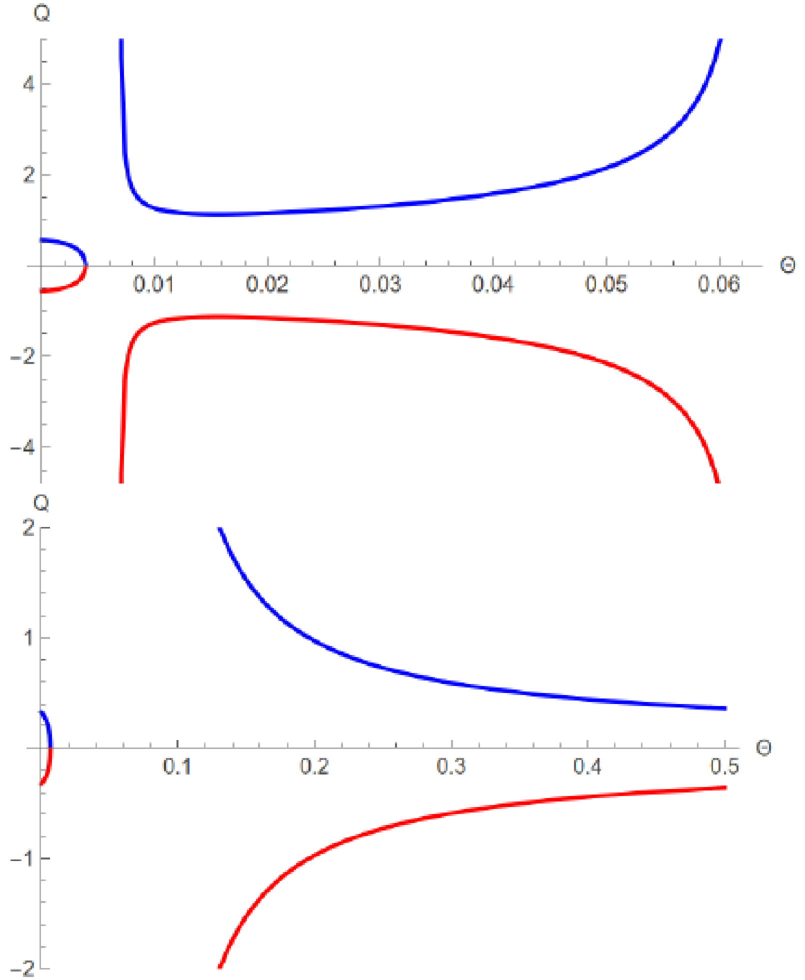
$$II:\frac{\partial^2 M}{\partial S^2}\left(\left(\frac{\partial^2 M}{\partial Q\partial\Theta}\right)^2-\frac{\partial^2 M}{\partial Q^2}\frac{\partial^2 M}{\partial\Theta^2}\right)=0$$

$$III:\frac{\partial^2 M}{\partial S^2}=\frac{\partial^2 M}{\partial S\partial Q}=\frac{\partial^2 M}{\partial S\partial\Theta}=0\ ,\quad \frac{\partial^2 M}{\partial S\partial Q}=\frac{\partial^2 M}{\partial Q^2}=0\ ,\quad \frac{\partial^2 M}{\partial S\partial\Theta}=\frac{\partial^2 M}{\partial\Theta^2}=0\ ,\quad \frac{\partial^2 M}{\partial Q^2}=\frac{\partial^2 M}{\partial\Theta^2}=0$$

Applying the mass (IX) we find the phase transition curves

$$I : Q^{(I)} = \pm S^{3/2}\sqrt{\frac{\sqrt{S} - 16\sqrt{\Theta}}{\pi(\sqrt{S} - 4\sqrt{\Theta})(\sqrt{S} - 12\sqrt{\Theta})}}$$

$$II \equiv III : Q^{(II)} = \pm S\sqrt{\frac{\sqrt{S} - 12\sqrt{\Theta}}{3\pi(\sqrt{S} - 4\sqrt{\Theta})^3}}$$



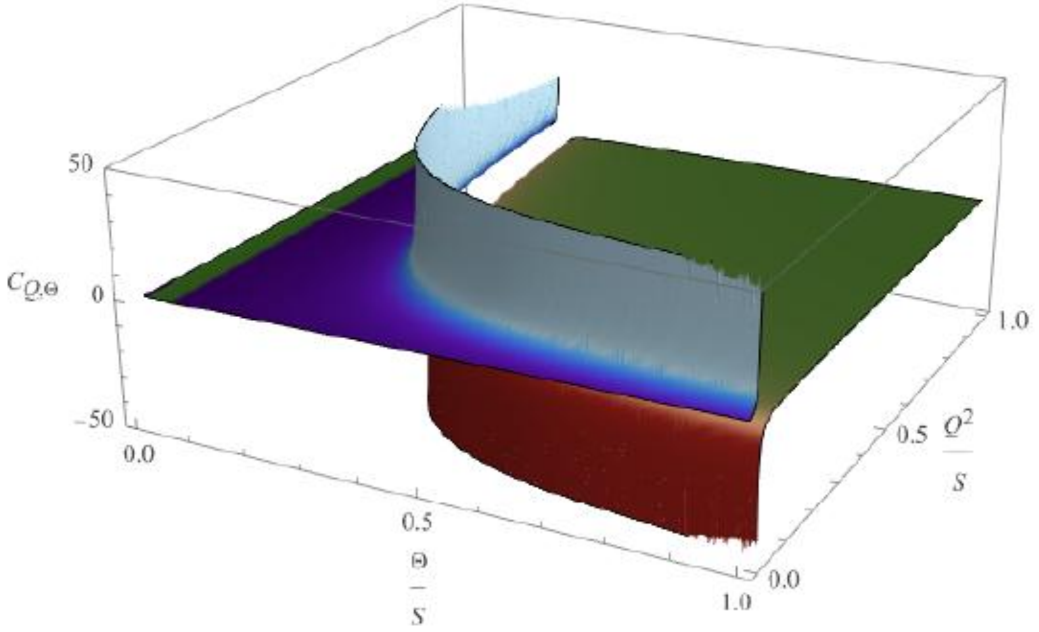
which recover the well-known NC Schwarzschild phase transition points $S = 144\Theta, 256\Theta$ [5] and the classical case $Q_{RN}^{(I)} = \pm\sqrt{\frac{S}{\pi}}$ and $Q_{RN}^{(II)} = \pm\frac{Q_{RN}^{(I)}}{\sqrt{3}}$

Heat capacity

$$C_{Q,\Theta} \equiv T\left(\frac{\partial S}{\partial T}\right) \equiv \frac{\frac{\partial M}{\partial S}}{\left(\frac{\partial^2 M}{\partial S^2}\right)} = \frac{2S(4\sqrt{\Theta}-\sqrt{S})\left(\sqrt{S}(\sqrt{S}-8\sqrt{\Theta})(S-\pi Q^2)-16\pi Q^2\Theta\right)}{S(12\sqrt{\Theta}-\sqrt{S})(S-3\pi Q^2)-48\pi Q^2\Theta(-3\sqrt{S}+4\sqrt{\Theta})}$$

with the phase transition curve

$$\frac{Q^2}{S} = \frac{12\sqrt{\frac{\Theta}{S}}-1}{3\pi\left(-1+12\sqrt{\frac{\Theta}{S}}-48\frac{\Theta}{S}+64(\frac{\Theta}{S})^{3/2}\right)}$$



Conclusions and perspectives

-NC spacetimes using Lorentzian distributions in large radius regime seems to be well-behaved in terms of quasi-homogeneity.

-Phase transitions can be studied from singularities in the Ricci scalar of the GTD metrics.

-GTD allow to choose any potential and provide additional phase transition conditions beyond that provided by the heat capacity or non-invariant descriptions.

-The NCRN lead to phase transition curves that reproduce well the classical and NC Schwarzschild cases.

-Additional degrees of freedom should be incorporated, for example, Λ, parameters for nonlinear electrodynamics,…

-This formalism has been applied within cosmological contexts, in particular for the early universe.

Bibliography

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