Hadronic Vacuum Polarization in lattice QCD

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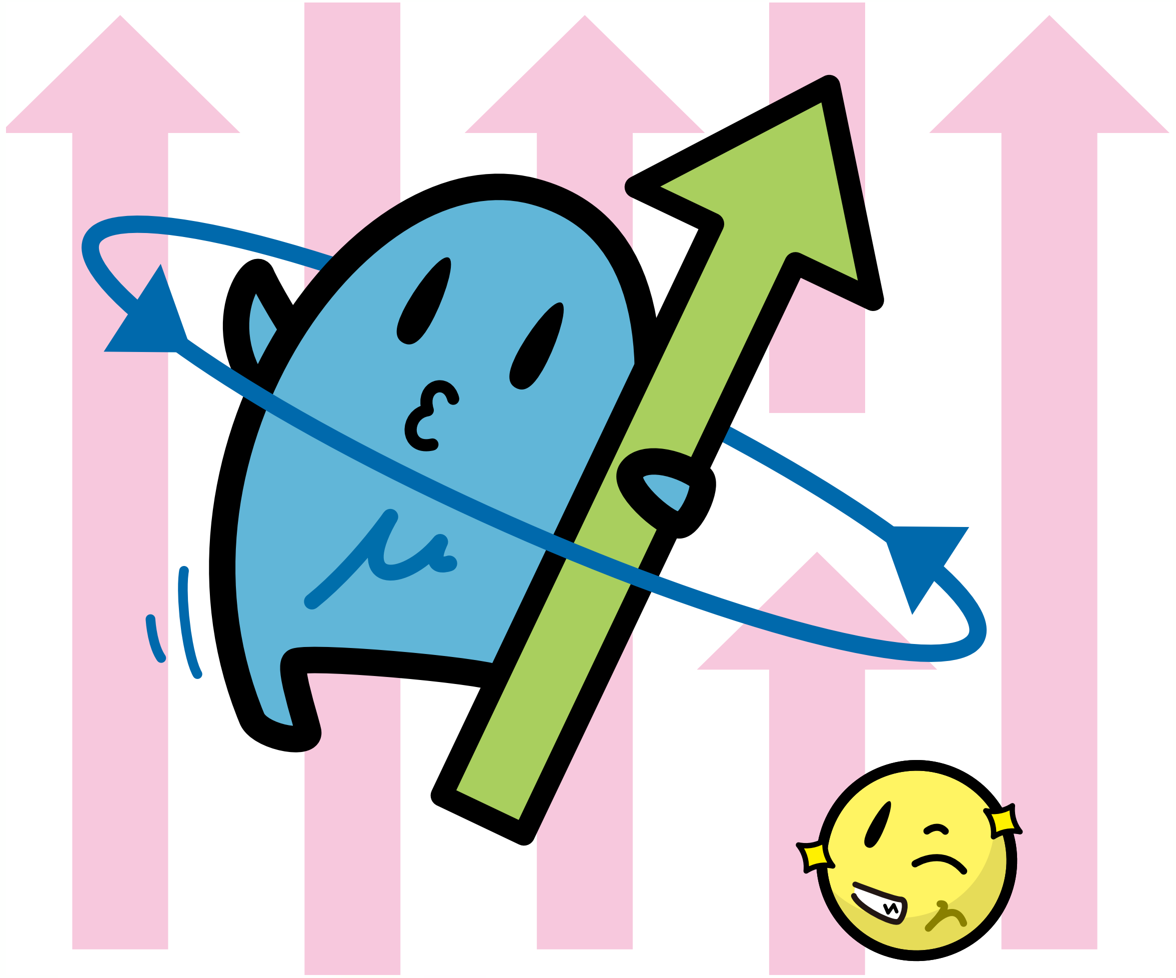
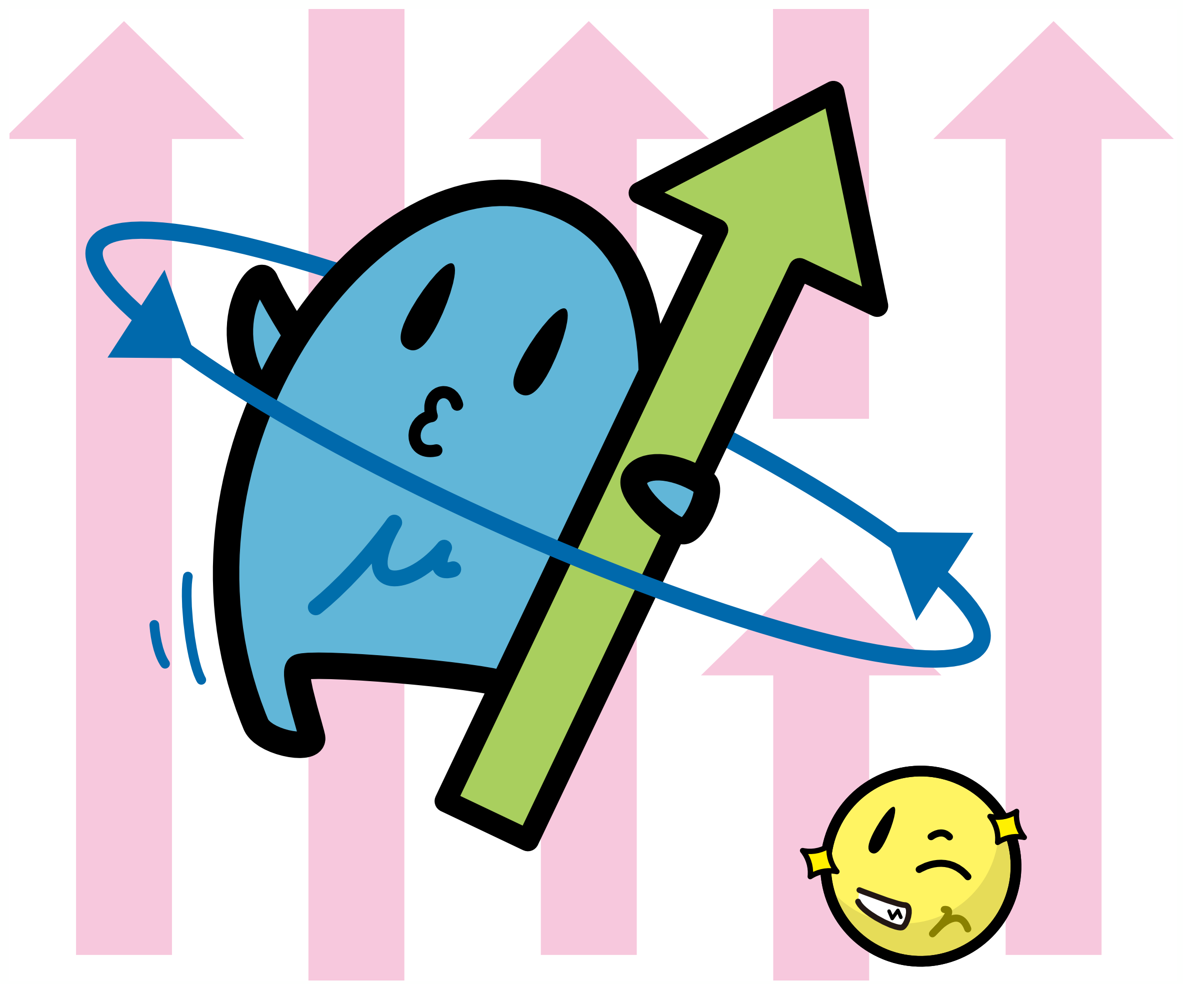
Simon Eidelman School on Magnetic Dipole Moments and Hadronic Effects
Nagoya University, 02-06 September 2024

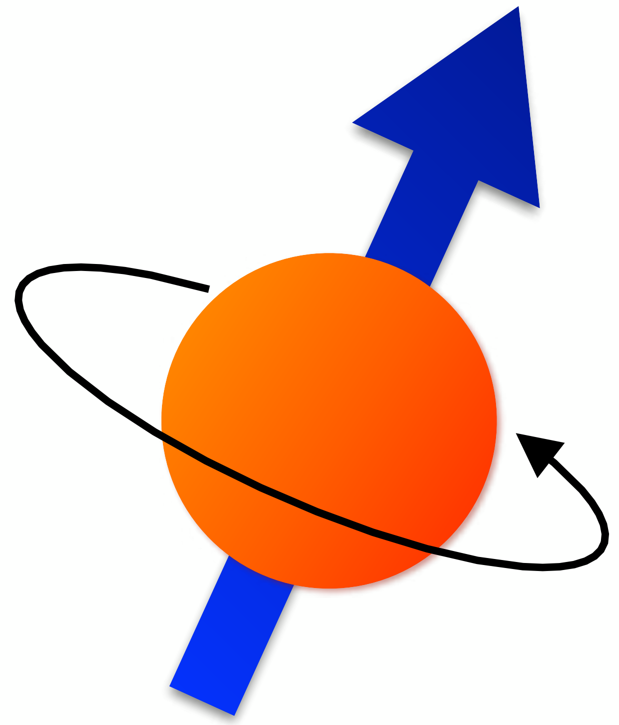


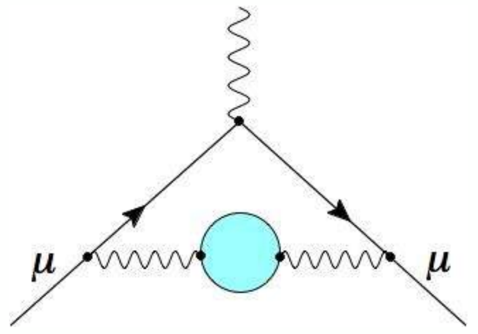
Kobayashi-Maskawa Institute for the Origin of Particles and the Universe
FlaP Flavor Physics International Research Center
科研費 KAKENHI
22K21347, 22K21347, 22K21350
WILHELM UND ELSE HERAEUS-STIFTUNG

Outline



- Introduction
- Introduction to lattice QCD
  - sources of systematic errors
  - state of the art
- Lattice HVP
  - Introduction
  - systematic errors
  - separation prescription
  - Finite Volume (FV) effects
  - long-distance tail
  - windows in Euclidean time
  - Results and updates from Lattice 2024
- Summary and Outlook

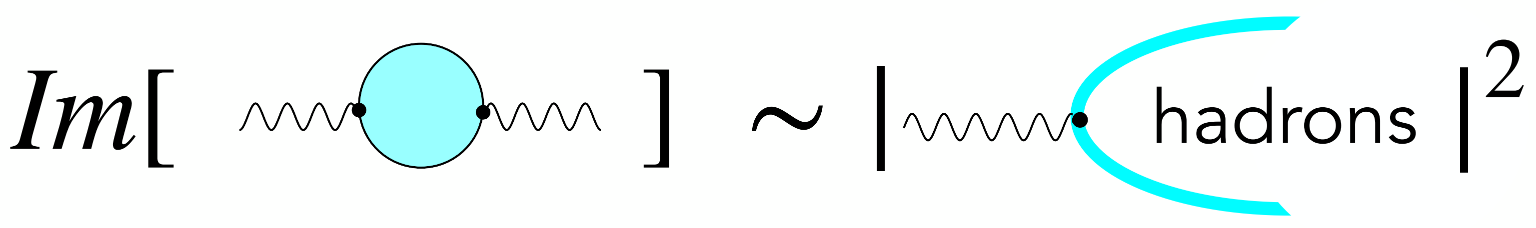




Hadronic Vacuum Polarization

- Dispersive, data-driven:
  HVP: integrate hadronic cross section over CM energy:

➠ Martin Hoferichter, Zhiqing Zhang

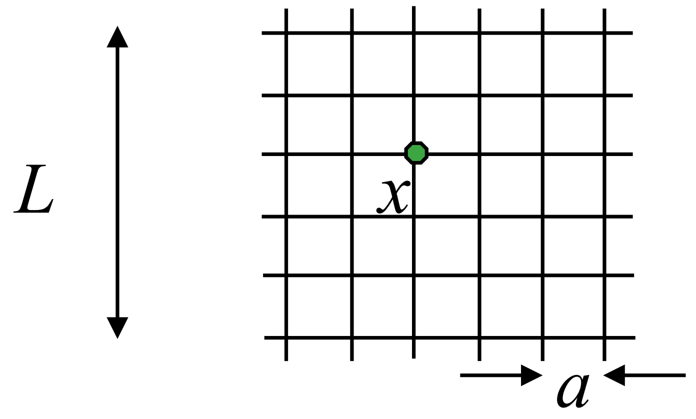


➠ $$a_\mu^{\rm HVP,LO} = \left(\frac{\alpha}{\pi}\right)^2 \int dq^2 \omega(q^2)\,\hat{\Pi}(q^2) = \frac{m_\mu^2}{12\pi^3}\int ds\,\frac{\hat{K}(s)}{s}\,\sigma_{\rm exp}(s)$$

Many experiments (over 20+ years) have measured the e^+e^-cross sections for (almost) all channels over the needed energy range with increasing precision.

- Direct calculation using Euclidean Lattice QCD

➠ $$a_\mu^{\rm HVP,LO} = 4\,\alpha^2 \int_0^\infty dt\,C(t)\,\tilde{w}(t)$$



Approximations:
discrete space-time (spacing a)
finite spatial volume (L), and time extent (T)
…

- *ab-initio* method to quantify QCD effects
- already used for simple hadronic quantities with high precision
- requires large-scale computational resources
- allows for entirely SM theory based evaluations

Integrals are evaluated numerically using Monte Carlo methods.

Muon g-2 Theory Initiative

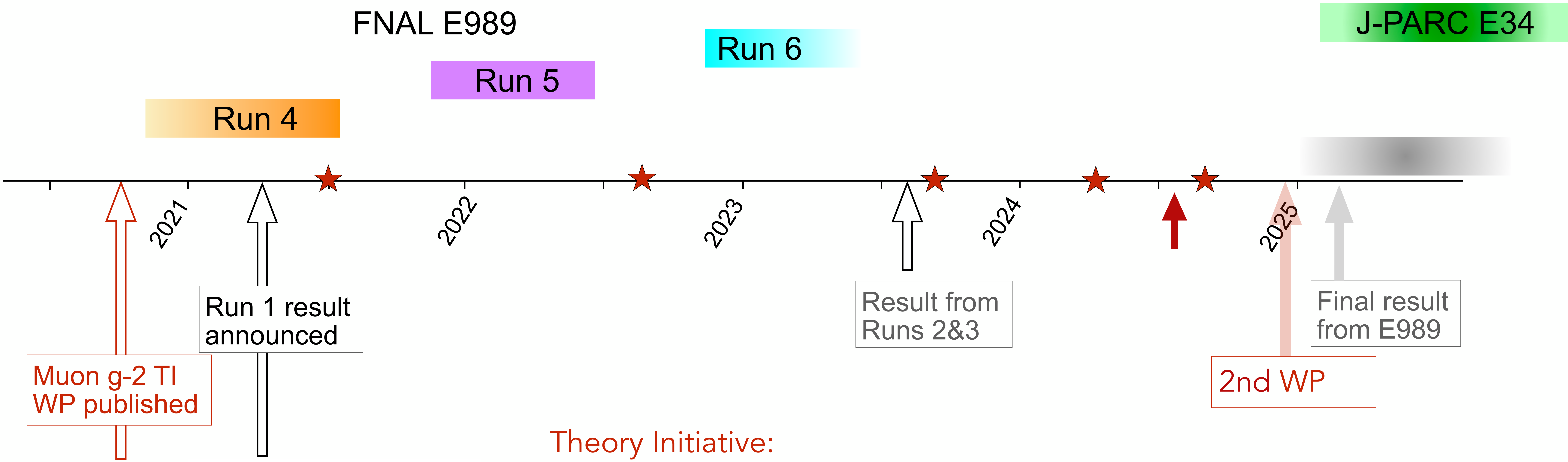
Steering Committee

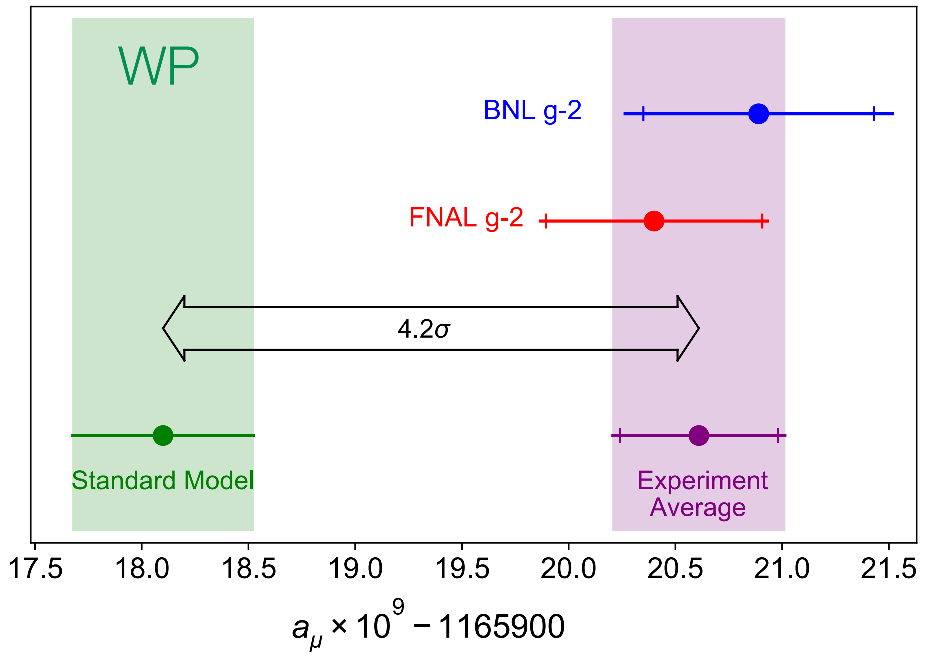
- Gilberto Colangelo (Bern)
- Michel Davier (Orsay) co-chair
- Aida El-Khadra (UIUC & Fermilab) **chair**
- Martin Hoferichter (Bern)
- Christoph Lehner (Regensburg University) co-chair
- Laurent Lellouch (Marseille)
- Tsutomu Mibe (KEK)
  J-PARC Muon g-2/EDM experiment
- Lee Roberts (Boston)
  Fermilab Muon g-2 experiment
- Thomas Teubner (Liverpool)
- Hartmut Wittig (Mainz)

https://muon-gm2-theory.illinois.edu

- Maximize the impact of the Fermilab and J-PARC experiments
  ➠ quantify and reduce the theoretical uncertainties on the SM prediction
- assess reliability of uncertainty estimates
- summarize the theory status: White Papers
- organize workshops to bring the different communities together:
  - First plenary workshop near Fermilab: 3-6 June 2017
  - …
  - Virtual Spring 2024 TI workshop hosted by UIUC: 15-17, 23-24 Apr 2024
  - Seventh plenary workshop hosted by KEK/KMI (Japan): 9-13 Sep 2024
  - Eight plenary workshop: France in Sep 2025
  - Ninth and tenth plenary workshops: US, UK

Near-term timeline





Theory Initiative:

- ☆ CMD-3 seminar (virtual): 27 March 2023 at 8:00am US CDT
- ☆ 2nd CMD-3 discussion meeting
- ☆ 8/9/2023: Status of Muon g-2 Theory in SM

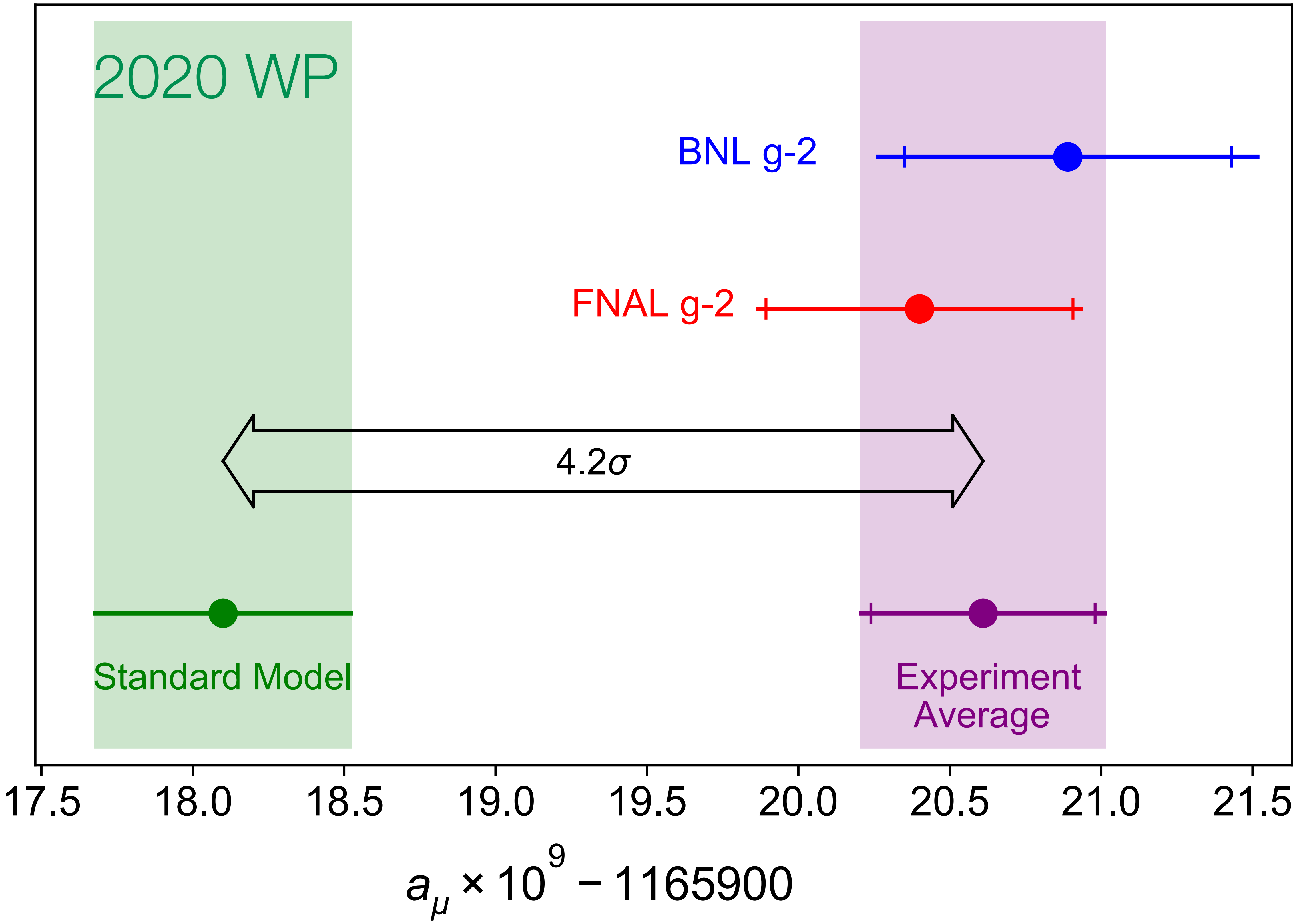
★ TI workshops:

Jun 2021 @ KEK (virtual) Sep 2023 @ Bern Sep 2024 @ KEK & KMI

Sep 2022 @ Higgscentre Apr 2024 (virtual)

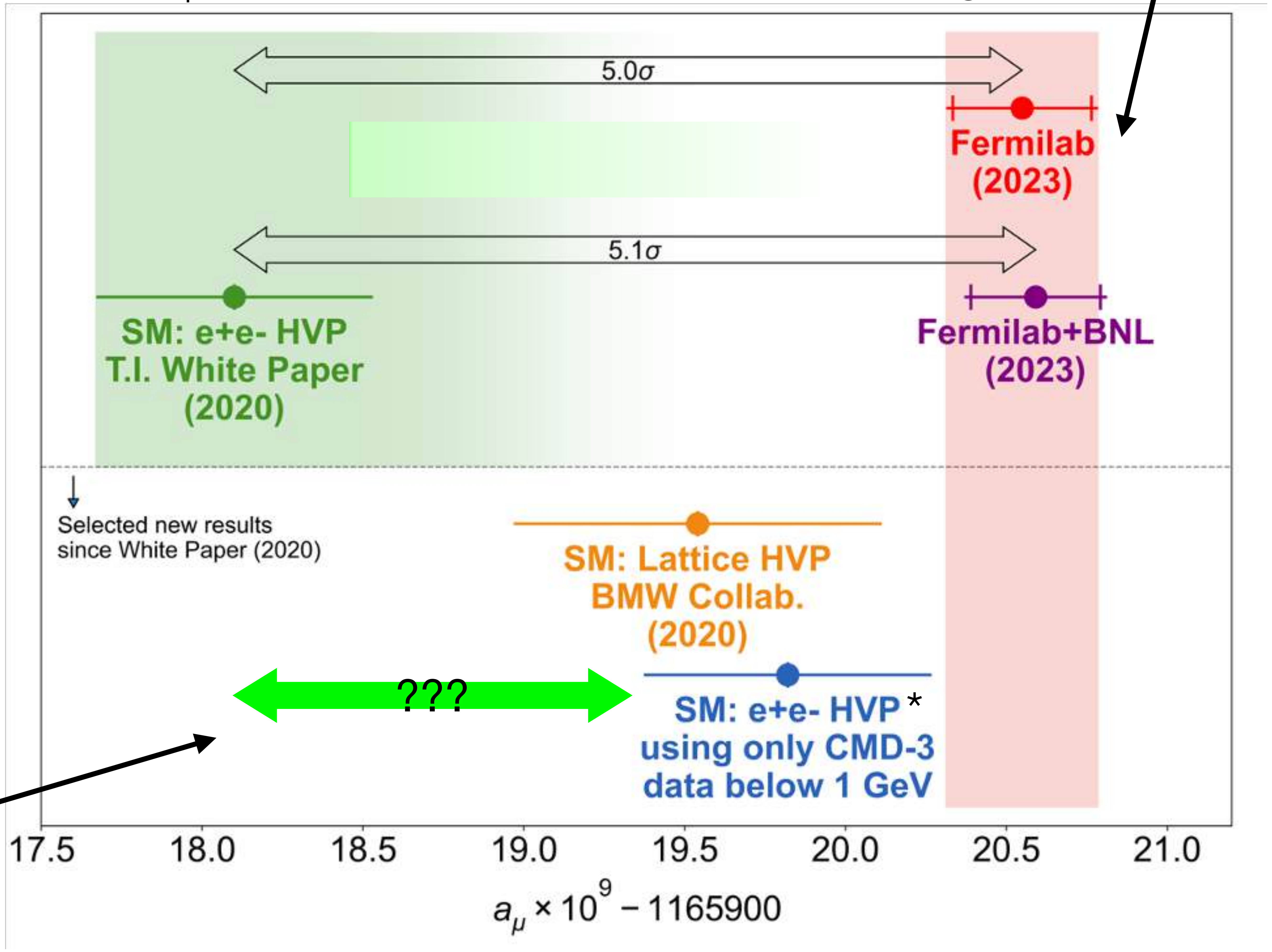
Experiment vs SM theory

- 2021



- 2023

adapted from J. Mott @ Scientific Seminar, 10 Aug 2023



Anna Driutti

Martin Hoferichter
Zhiqing Zhang

*A. Keshavarzi @ Lattice 2023

Experiment vs SM theory

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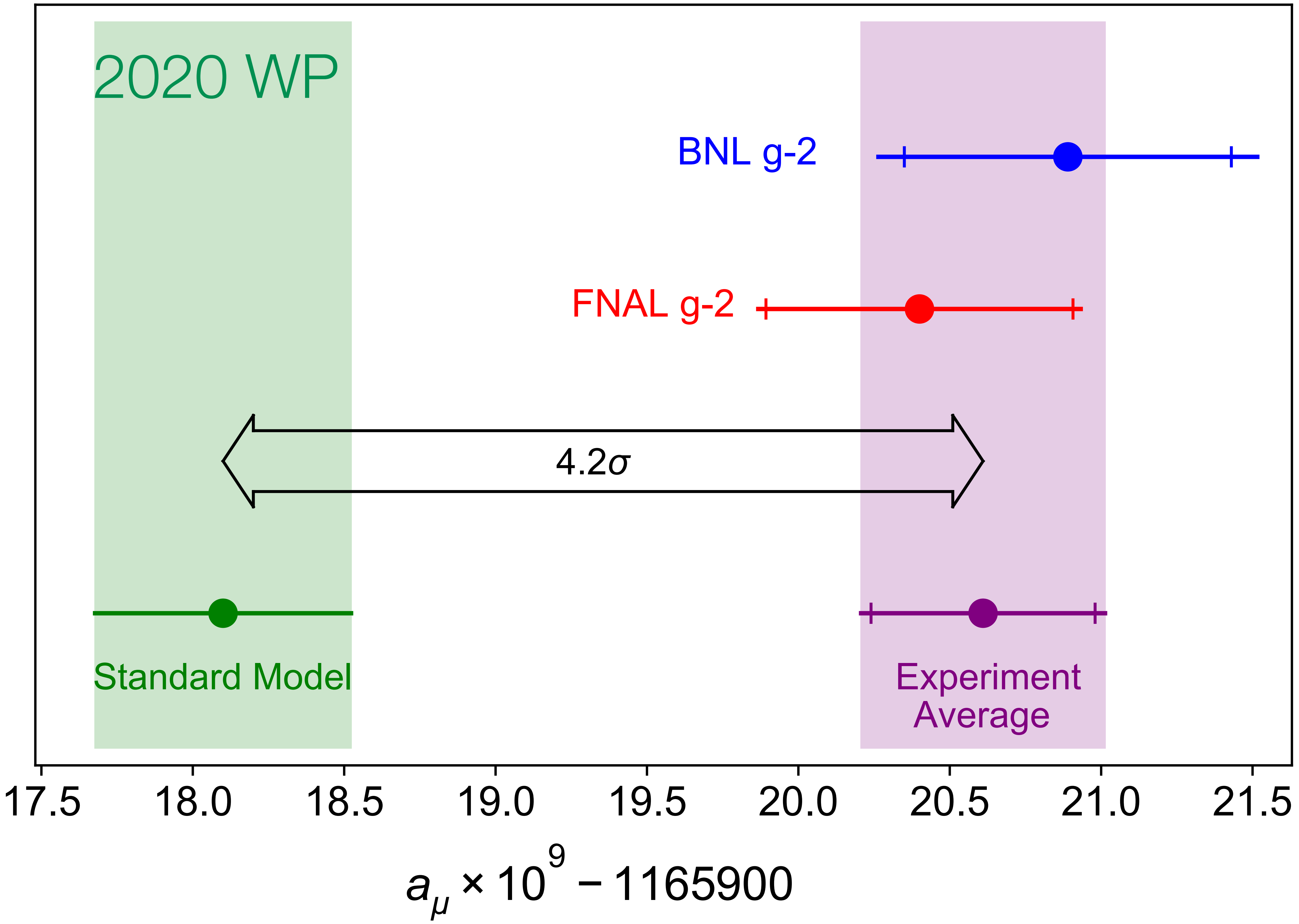

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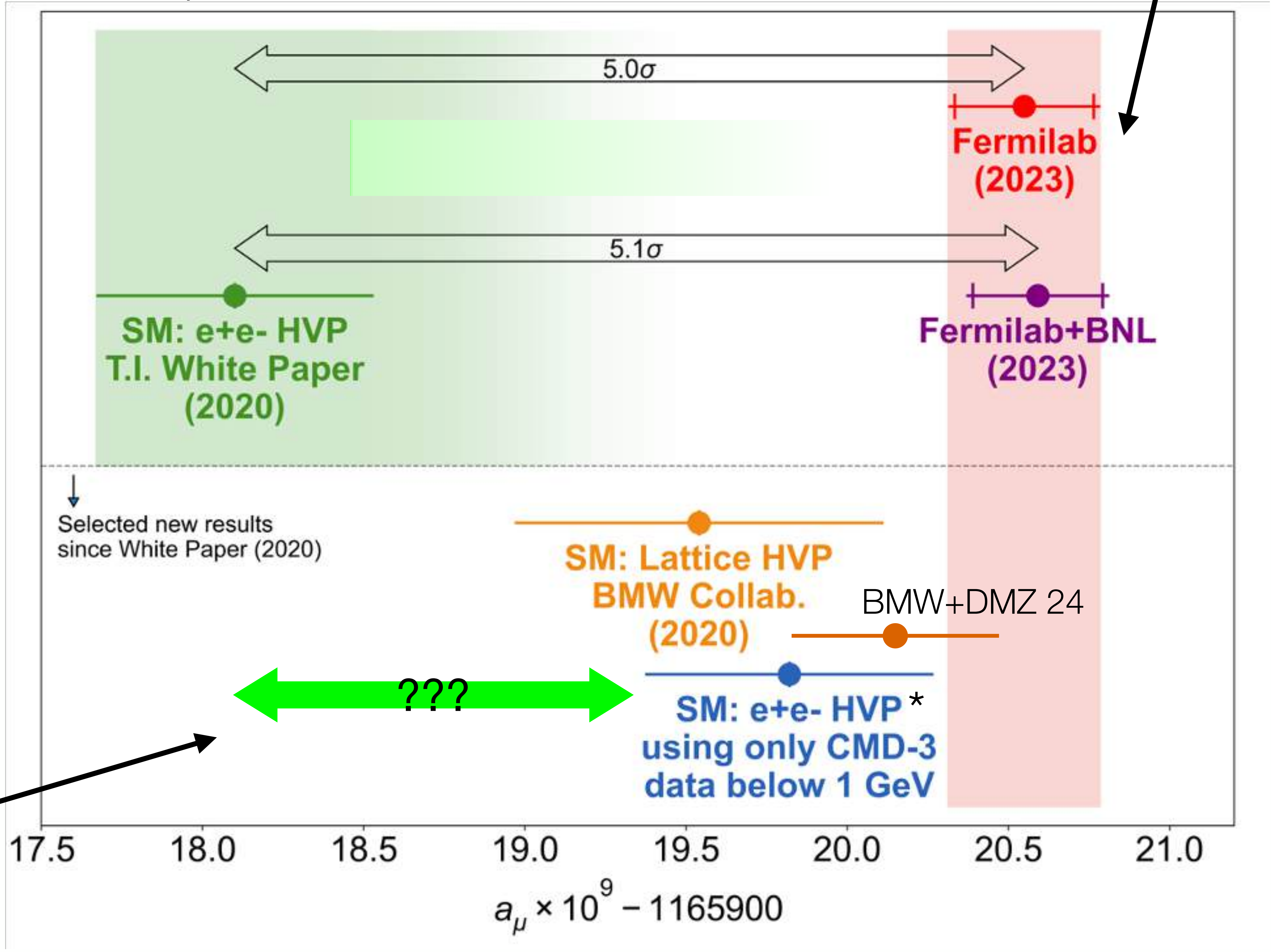

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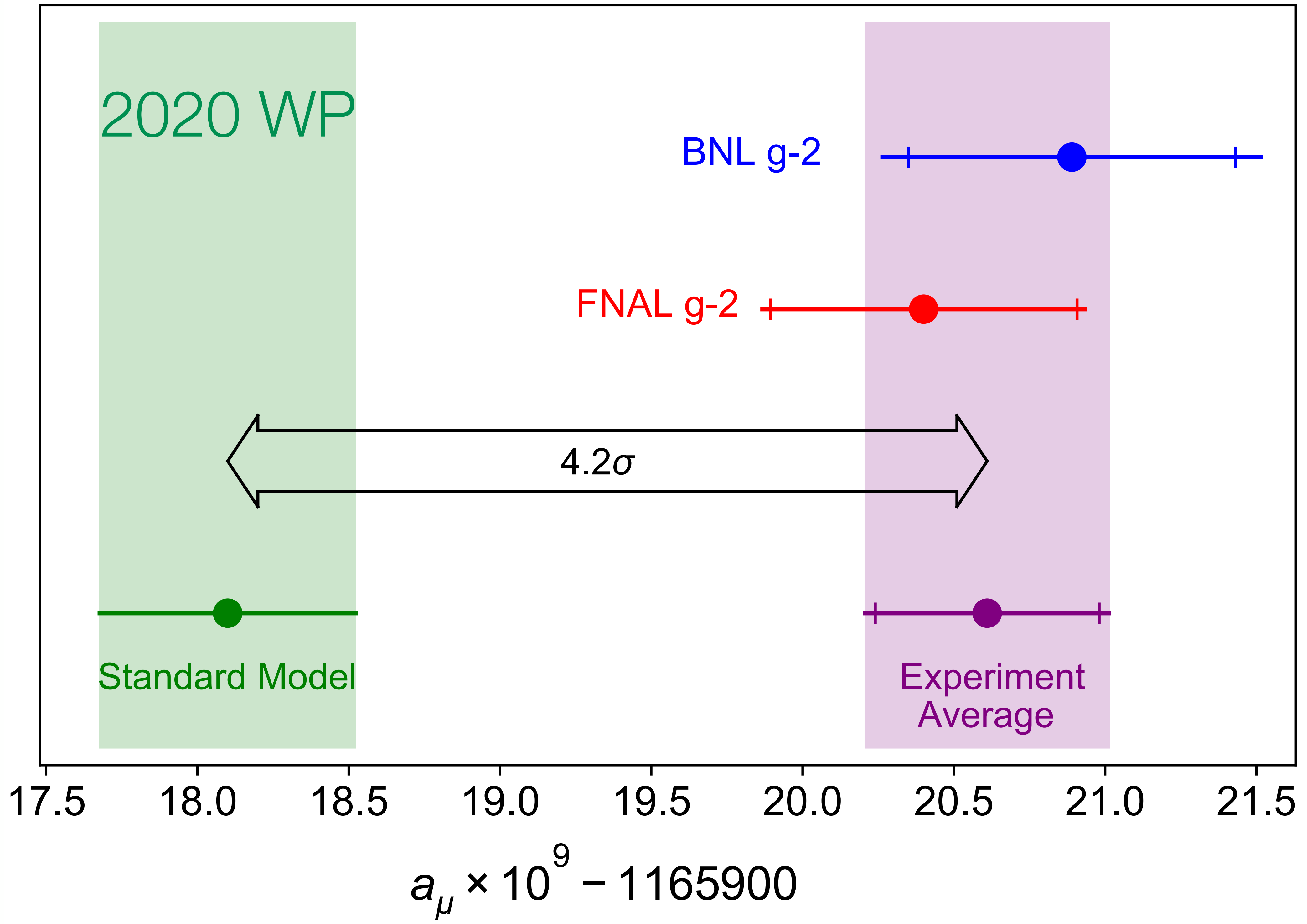
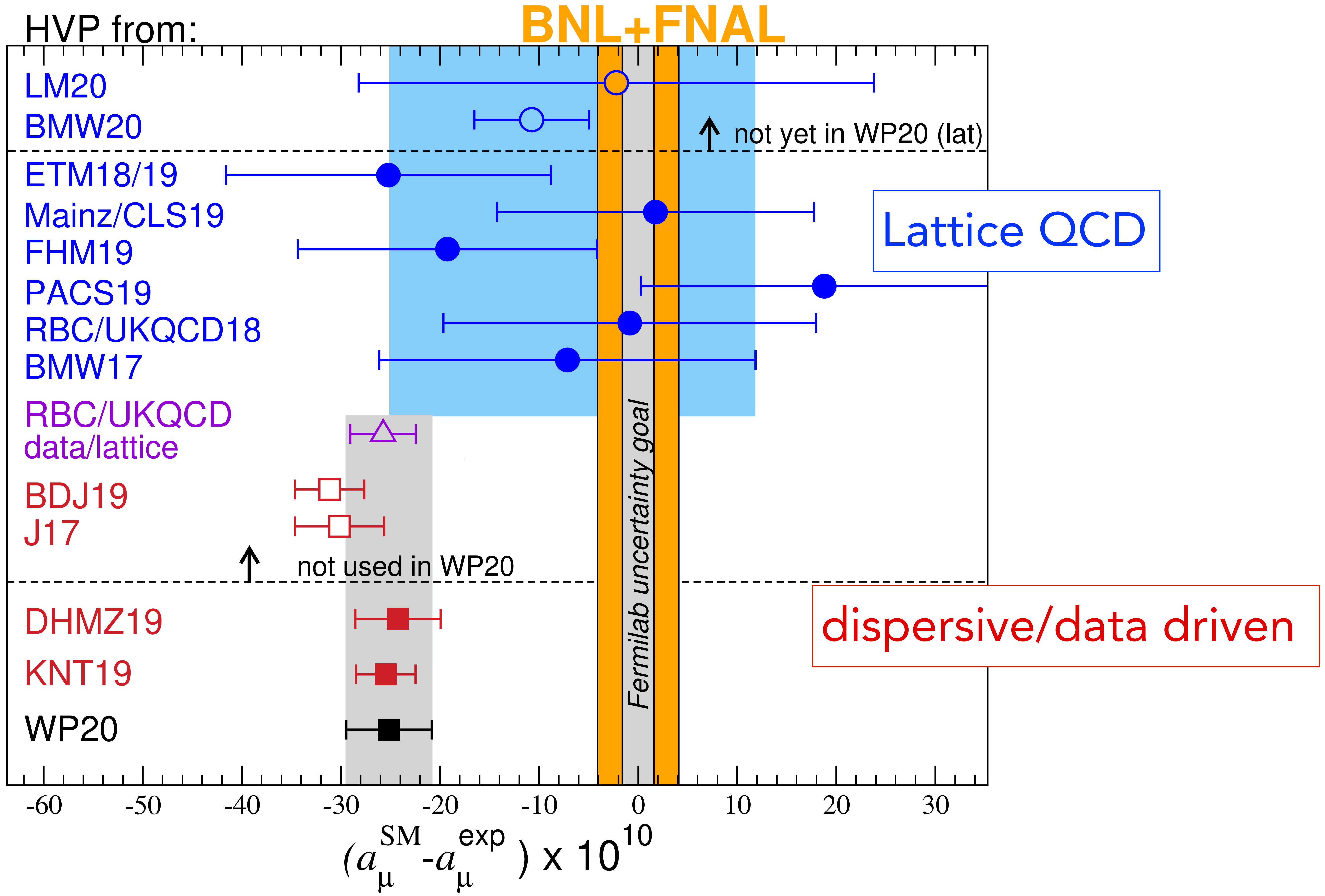
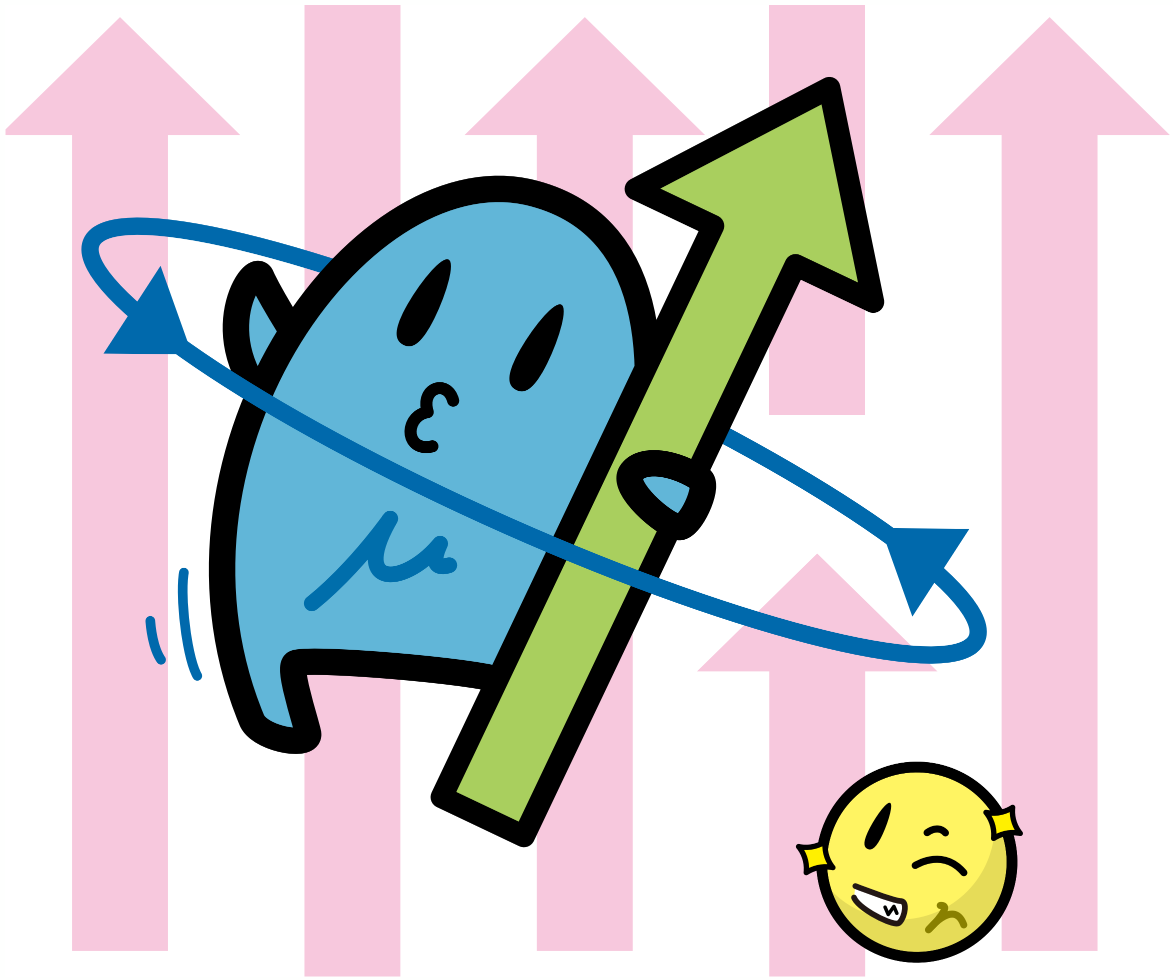




$$a_\mu^{\rm SM} \quad \langle \quad a_\mu^{\rm HVP} + \left[a_\mu^{\rm QED} + a_\mu^{\rm Weak} + a_\mu^{\rm HLbL}\right]$$

Outline



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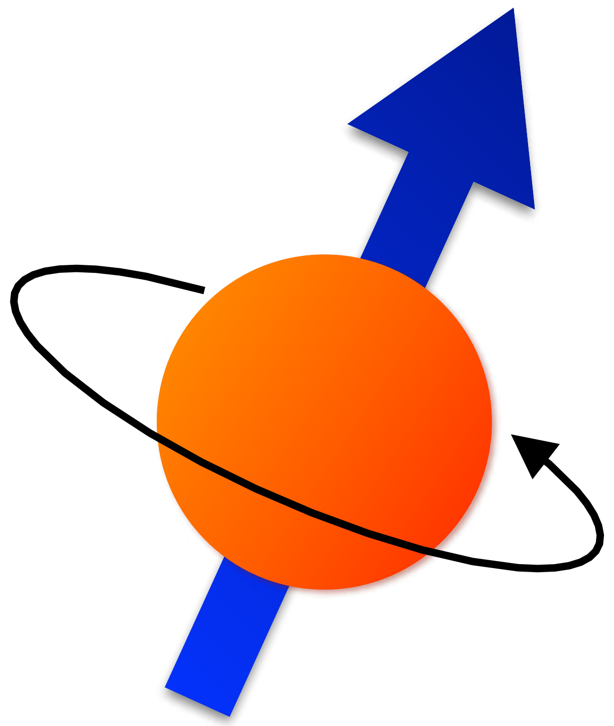


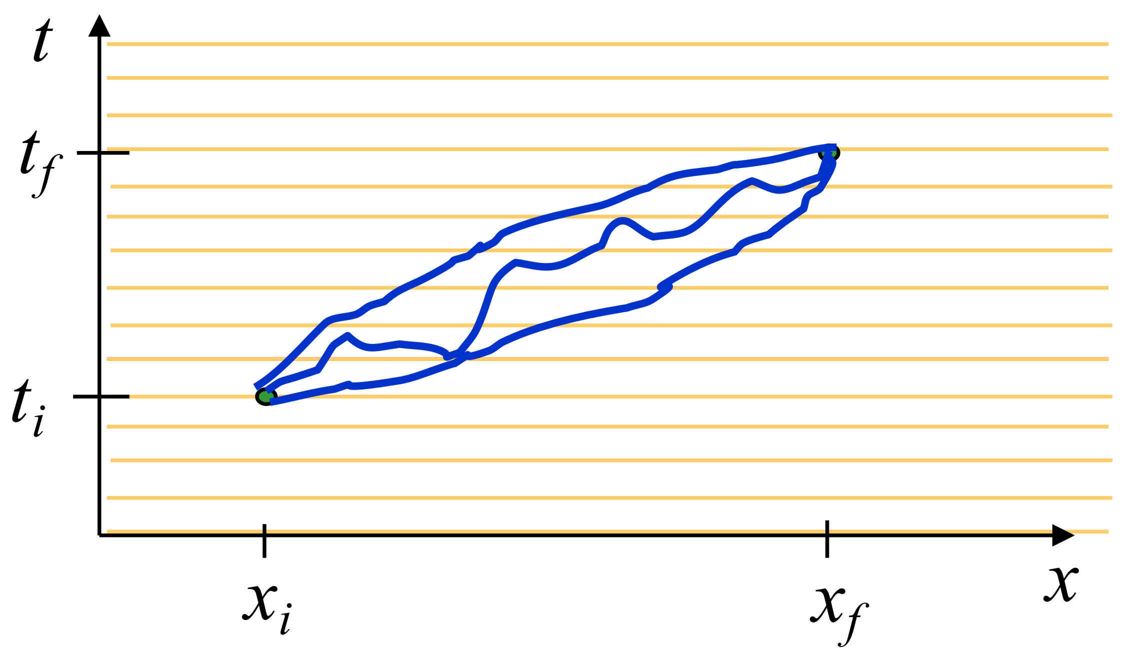
Lattice Quantum Field Theory

Feynman's Path Integral in Quantum Mechanics

$$\langle x_f | \, e^{-iH(t_f - t_i)} \, | x_i \rangle = \int \mathscr{D}x(t) \, e^{iS}$$

$x_k = x(t_k), \; x_0 = x_i, \, x_N = x_f$

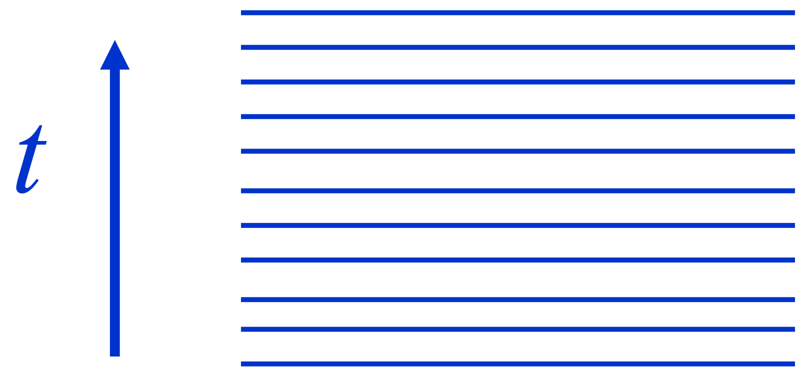
$$\int \mathscr{D}x(t) = \lim_{N \to \infty} \prod_{k=0}^{N} \int dx_k$$

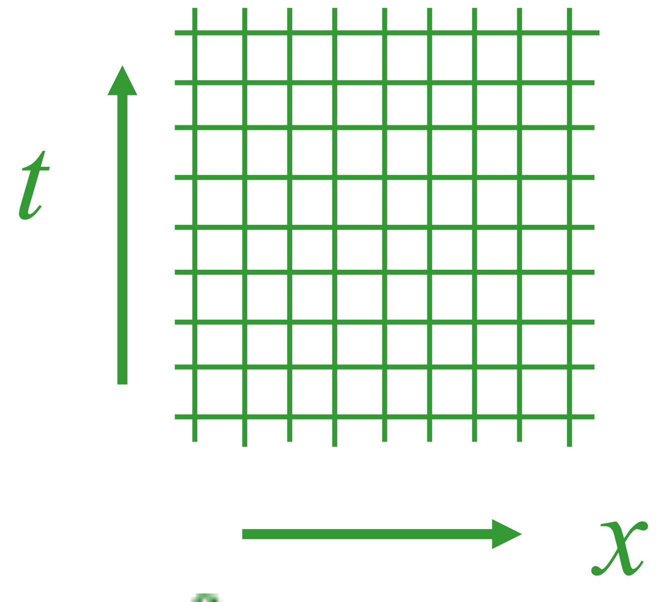


For efficient numerical computation:
use Euclidean time $t \to it$ so that $e^{iS} \to e^{-S}$

[see also Creutz & Freedman, Annals Phys. 132 (1981) 427;
G.P. Lepage, hep-lat/0506036]

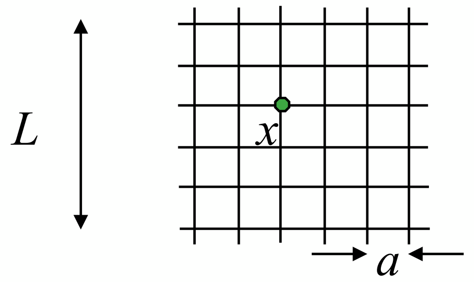
QM	QFT
$x(t)$	$\phi(x)$
time lattice	space-time lattice





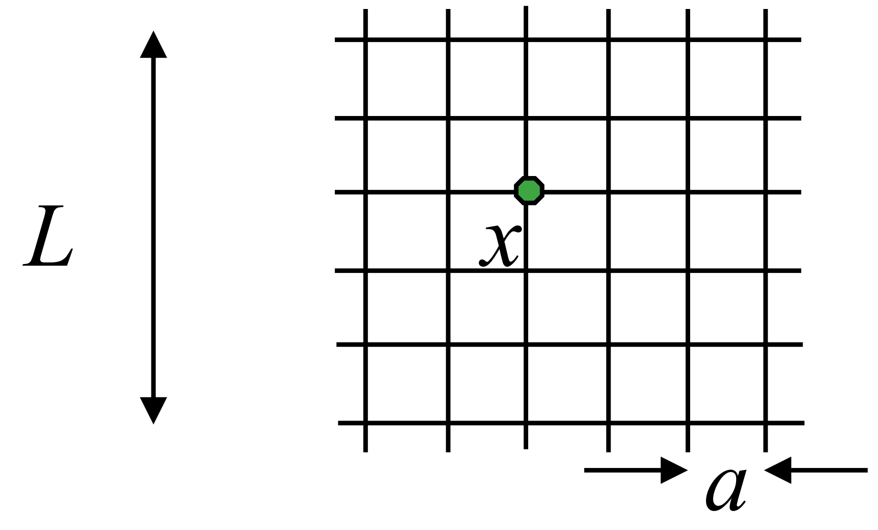
QM	QFT
$\int \mathcal{D}x(t)$	$\int \mathcal{D}\,\phi(x)$
N dimensions	N^4 dimensions

Integrals are evaluated numerically using Monte Carlo methods.

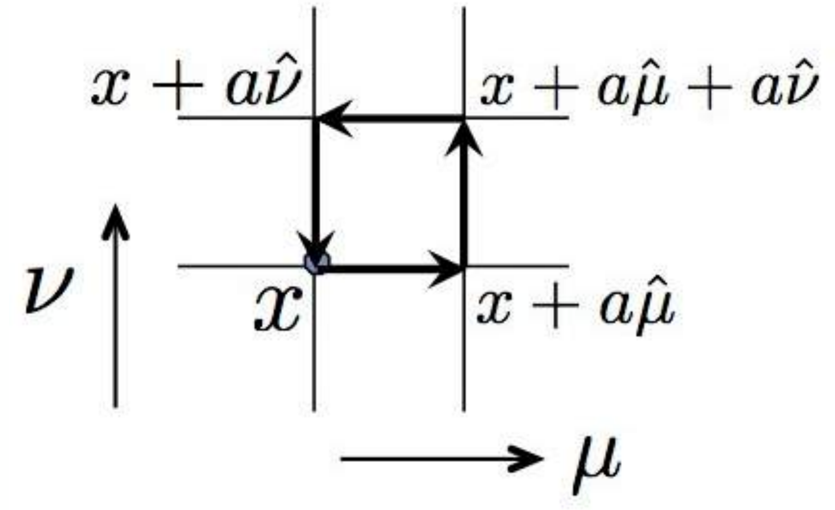


Lattice QCD Introduction

$$\mathcal{L}_{\mathrm{QCD}} = \sum_f \bar\psi_f (\not{D} + m_f)\psi_f + \frac{1}{4}\mathrm{tr} F_{\mu\nu}F^{\mu\nu}$$



- discrete Euclidean space-time (spacing a)
- finite spatial volume (L)
- finite time extent (T)



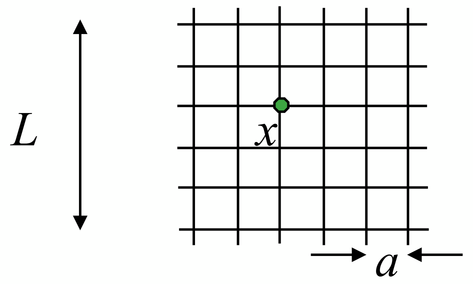
- discrete space-time:
  derivatives ➠ difference operators
  covariant ➠ connected with link fields
- computer memory and storage are finite:
  ➠ finite time extend and finite volume
- numerical integration with Monte Carlo (importance sampling): ➠ statistical errors

Wilson [Phys.Rev. D10 (1974) 2445-2459]:

- keep gauge invariance manifest on space-time lattice using link fields: $U_\mu(x) = U(x, x + a\hat\mu) = \mathrm{P}\, e^{ig\int_x^{x+a\hat\mu} dx'\cdot A(x')}$
  ➠ construct gluon action using gauge invariant Wilson loops
- Wilson gauge action built using plaquettes:

$$S_g = \frac{1}{g^2}\sum_{\mu,\nu,x} P_{\mu\nu}(x)$$

$$P_{\mu\nu} = \mathrm{ReTr}[1 - U_\mu(x)U_\nu(x + a\hat\mu)U_\mu^\dagger(x + a\nu)U_\nu^\dagger(x)]$$



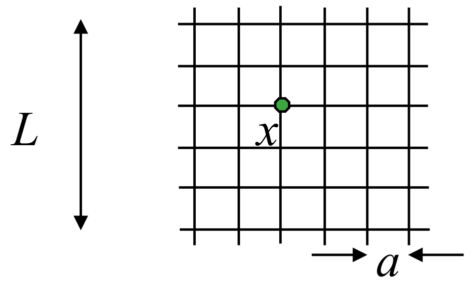
Lattice QCD Introduction: quark discretizations

Fermion doubling problem ⇔ chiral symmetry

- Staggered quarks (a.k.a Kogut-Susskind)
  reduce the number of doublers (staggering) but keep some (a.k.a tastes)
  dominant discretization effects due to taste-breaking effects (can be corrected analytically) ➠ $O(a^2)$
  improved actions with discretization + TB effects starting at ~ $O(\alpha_s a^2, \alpha_s^2 a^2)$ (Astad, HISQ)
  various smearing schemes to reduce taste-breaking effects
  computationally inexpensive
- (improved) Wilson quarks
  no doublers, but chiral symmetry broken explicitly ➠ $O(a)$ discretization effects
  remove $O(a)$ effects nonperturbatively (NP improved, twisted mass, …) ➠ $O(a^2)$
  moderate computational cost
- Domain wall quarks (live in 5 dimensions)
  no doublers, chiral symmetry breaking suppressed by length of 5th dim. ➠ $O(a^2)$
  small discretization effects
  high computational cost
- …

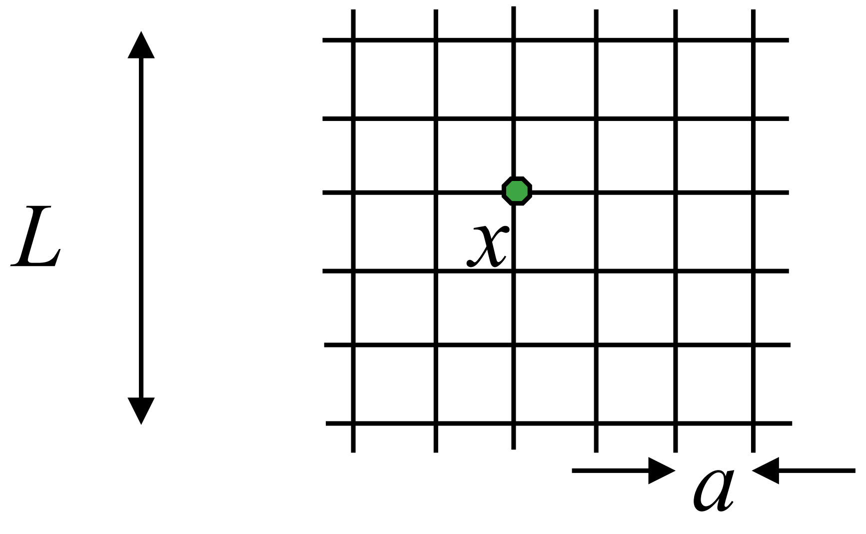
considerations for b quarks: ➠ appendix

- new ideas: workshop on novel fermion actions https://indico.mitp.uni-mainz.de/event/314/



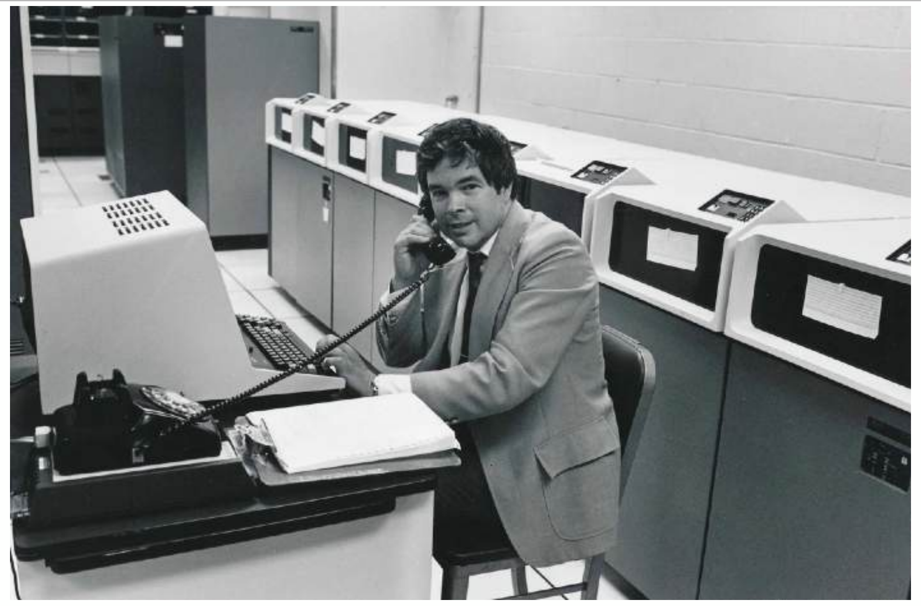
Lattice QCD Introduction

$$\mathcal{L}_{\rm QCD} = \sum_f \bar\psi_f (\not{D} + m_f)\psi_f + \frac{1}{4}{\rm tr} F_{\mu\nu}F^{\mu\nu}$$

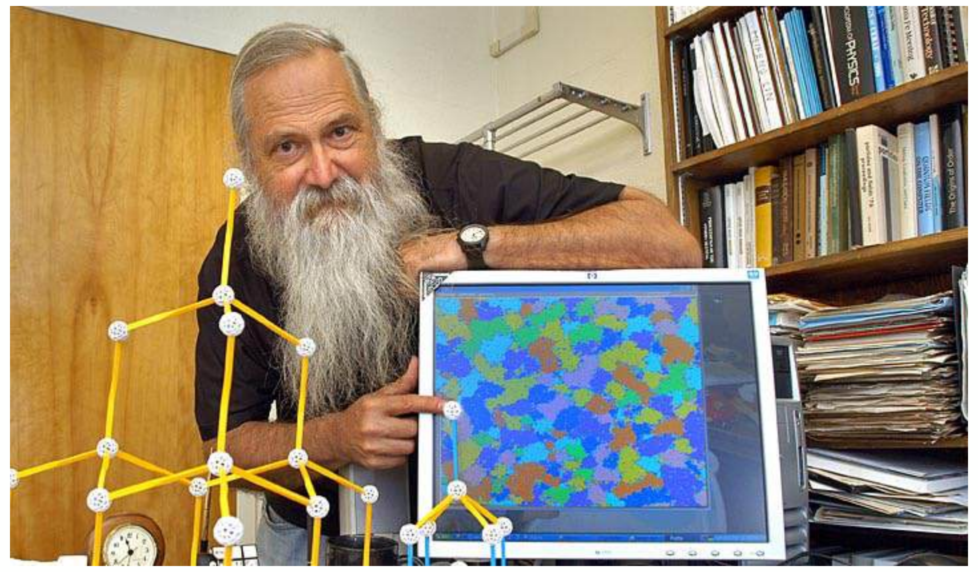


- discrete Euclidean space-time (spacing a)
  derivatives ➙ difference operators, etc…
- finite spatial volume (L)
- finite time extent (T)

Ken Wilson

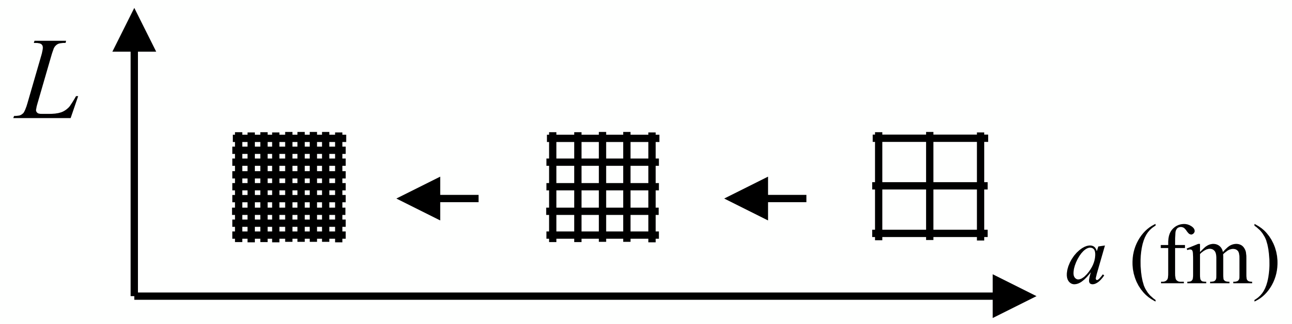


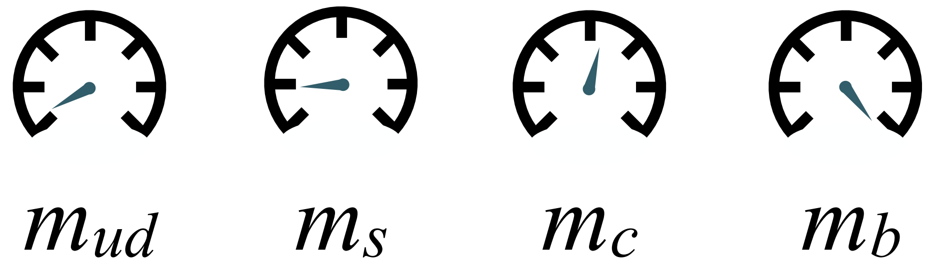
Mike Creutz

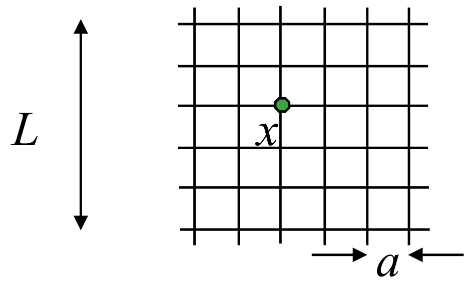


adjustable parameters

- lattice spacing: $a \to 0$
- finite volume, time: $L \to \infty$, $T > L$
- quark masses (m_f): $M_{H,\rm lat} = M_{H,\rm exp}$
  tune using hadron masses
  extrapolations/interpolations
  $m_f \to m_{f,\rm phys}$



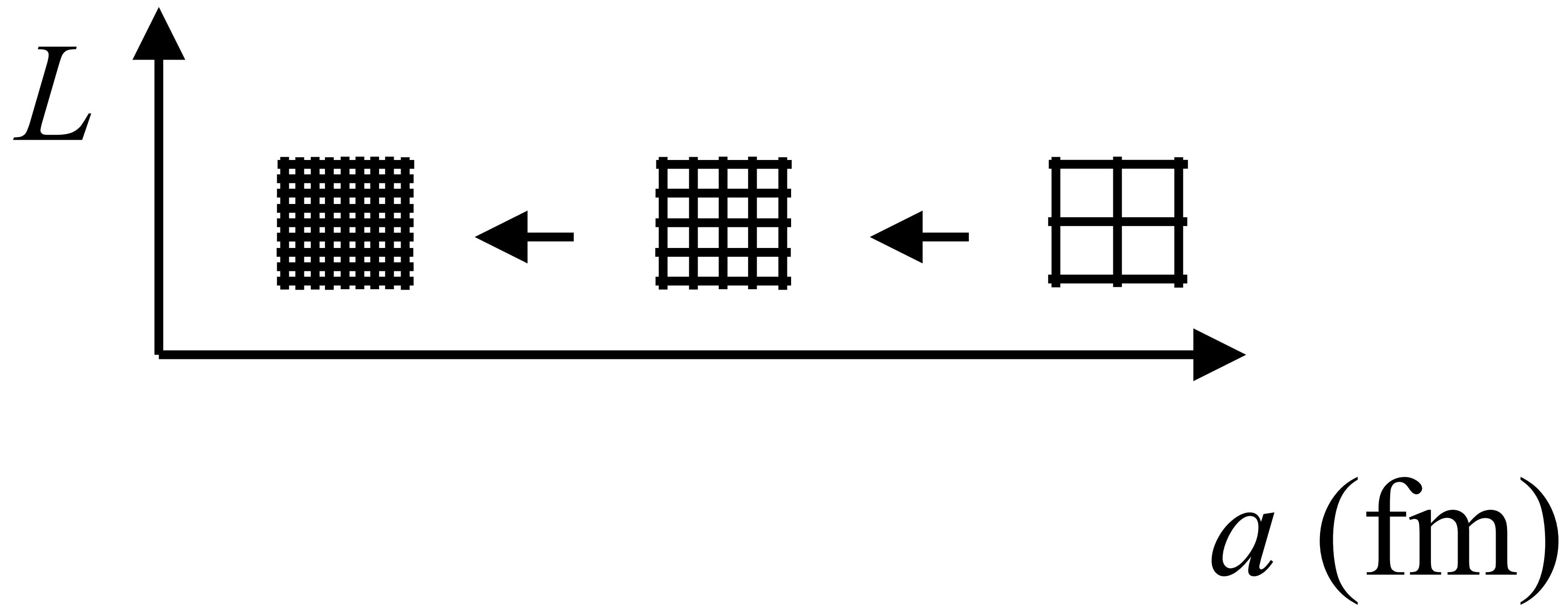


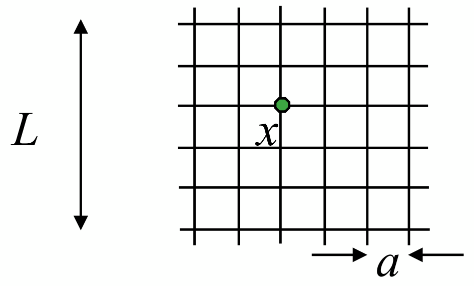


Lattice QCD Introduction

discretization effects — continuum extrapolation

- typical momentum scale of quarks gluons inside hadrons: $\sim\Lambda_{\rm QCD}$
- make a small to separate the scales: $\Lambda_{\rm QCD} \ll 1/a$
- Symanzik EFT: $\langle\mathcal{O}\rangle^{\rm lat} = \langle\mathcal{O}\rangle^{\rm cont} + O(a\Lambda)^n$, $n \geq 2$
  - provides functional form for extrapolation (depends on the details of the lattice action)
  - can be used to build improved lattice actions
  - can be used to anticipate the size of discretization effects



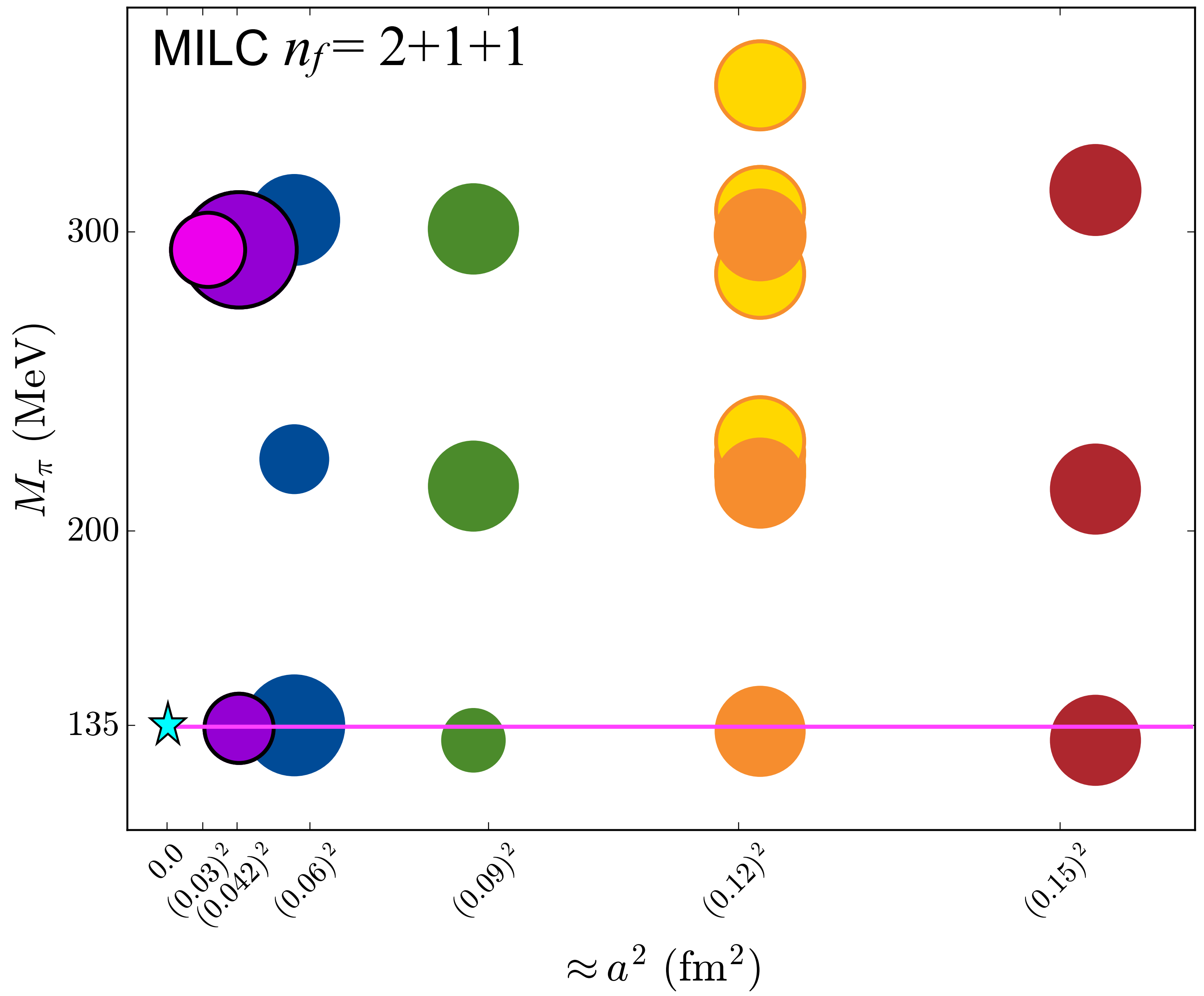


Lattice QCD Introduction

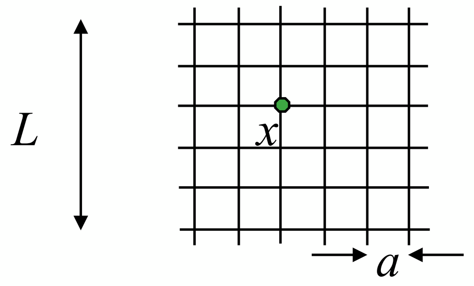
systematic error analysis

...of lattice spacing, chiral, heavy quark, and finite volume effects is based on Effective Field Theory (EFT) descriptions of QCD ➙ ab initio

- finite a: Symanzik EFT
- light quark masses: Chiral Perturbation Theory
- heavy quarks: HQET
- finite L: finite volume EFT



In practice:
stability and control over systematic errors depends on the lattice action(s) employed, underlying simulation parameters (available computational resources), analysis choices, …



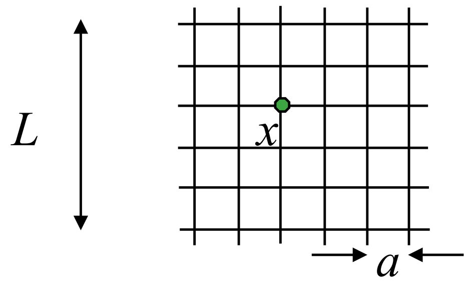
Lattice QCD Introduction

Steps of a lattice QCD calculation:

$$\langle \mathcal{O} \rangle \sim \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}U \, \mathcal{O}(\psi, \bar{\psi}, U) \, e^{-S} \qquad S = \int d^4x \left[\bar{\psi}(\not{D} + m)\psi + \frac{1}{4}(F^a_{\mu\nu})^2 \right]$$

Note: Integrate fermion fields by hand ➠ $\det(\not{D} + m)$ in integral. The operators $\mathcal{O}$ are then functions of $(\not{D}+m)^{-1}$ and gluon fields.

1. generate gauge-field configurations according to $\det(\not{D}+m)\, e^{-S}$
2. calculate quark propagators, $(\not{D}+m_q)^{-1}$ on each gauge-field configuration for each valence quark flavor and source point(s).
3. The hadronic correlation functions $\langle \mathcal{O} \rangle$ are obtained by tying together quark propagators into (usually 2, 3,4-pt functions)
4. statistical analysis to extract observables, such as hadron masses, energies, hadronic matrix elements, HVP,…. from correlation functions
5. systematic error analysis



Lattice QCD Introduction

The State of the Art

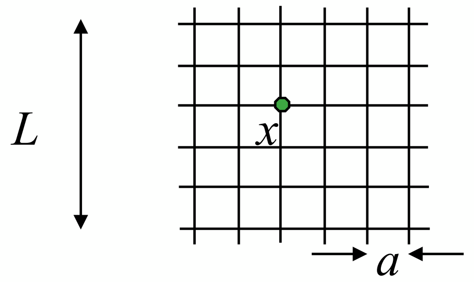
Lattice QCD calculations of simple quantities (with at most one stable meson in initial/final state) that **quantitatively account for all systematic effects** (discretization, finite volume, renormalization,…) in some cases with

- sub percent precision.
- total errors that are commensurate (or smaller) than corresponding experimental uncertainties.

Progress due to a virtuous cycle of theoretical developments, improved algorithms/methods and increases in computational resources (``Moore's law")

Scope of LQCD calculations is increasing due to continual development of new methods:

- nucleon matrix elements
- nonleptonic kaon decays ($K \to \pi\pi$, ϵ',…)
- resonances, scattering ($\pi\pi \to \rho$,…)
- long-distance effects (ΔM_K, …)
- QED corrections
- radiative decay rates
- structure: PDFs, GPDs, TMDs, …
- inclusive decay rates ($B \to X_c \ell \nu$,…)
- …



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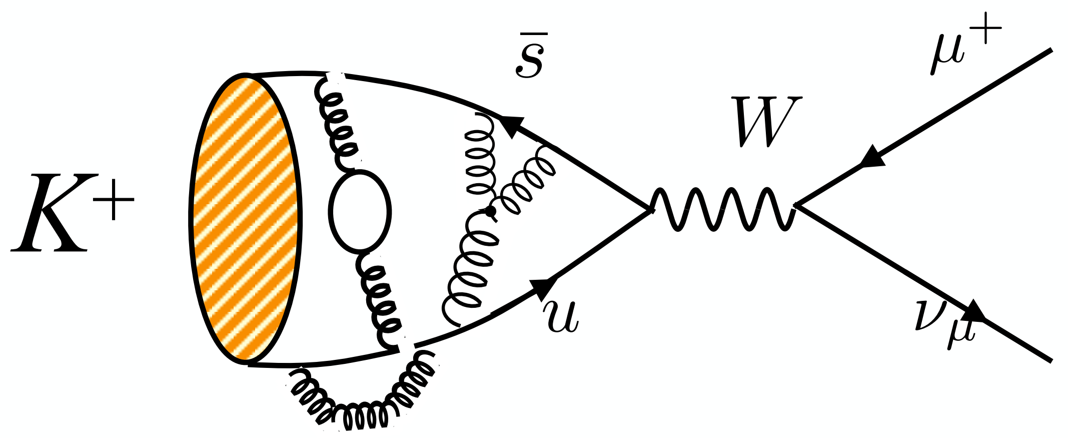
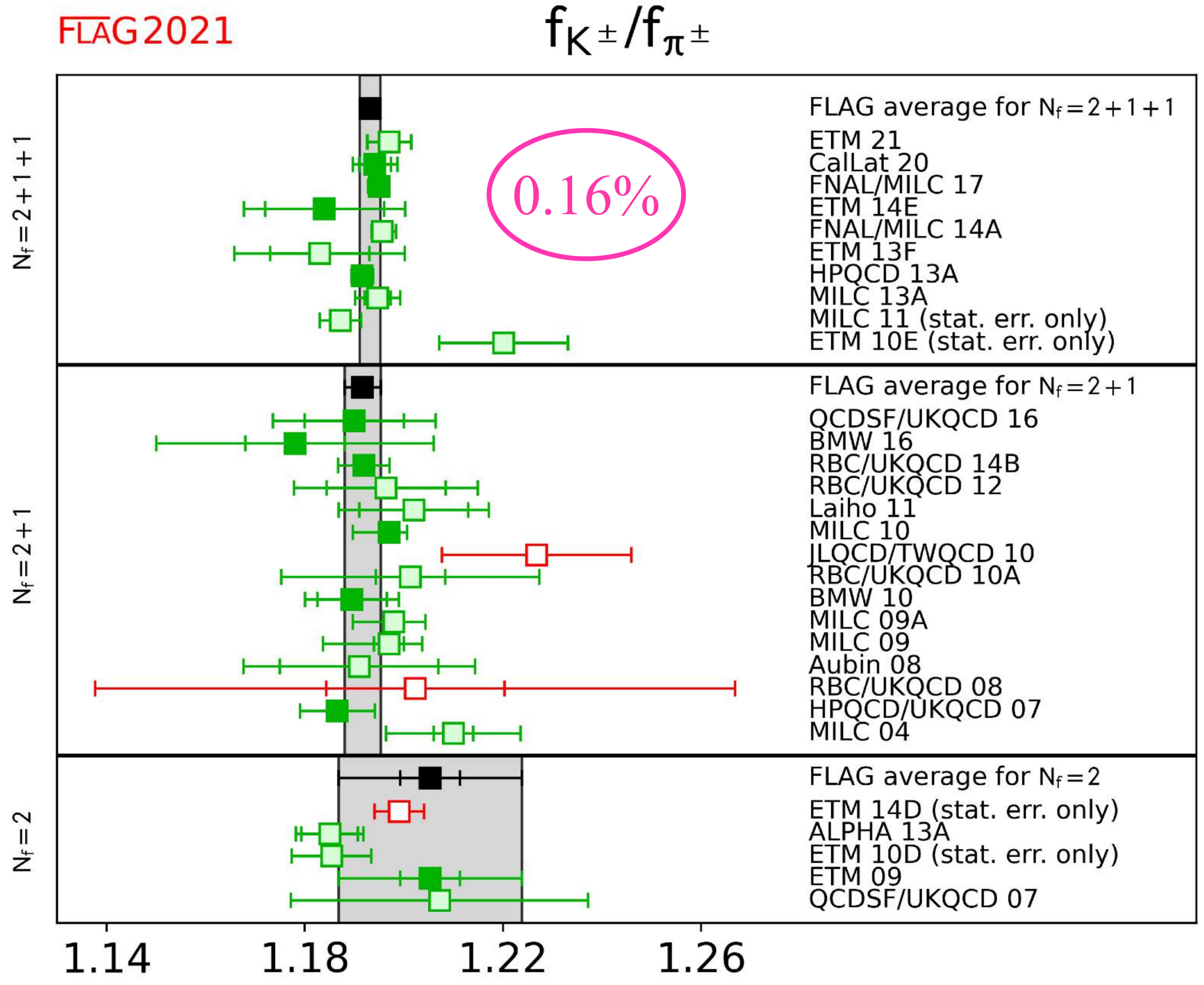




S. Aoki et al [FLAG 2021 review, arXiv:2111.09849, EPJC 2022]

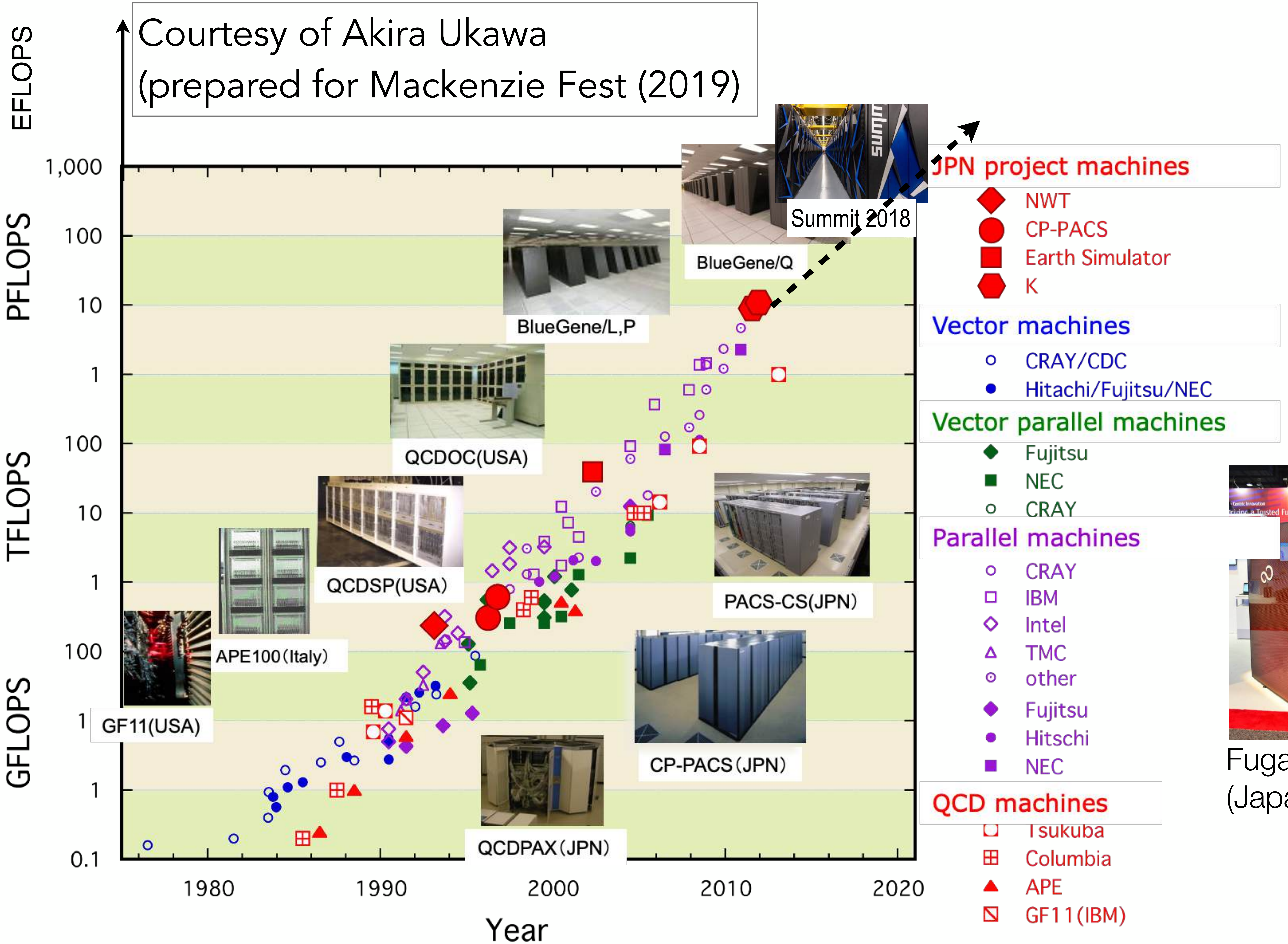
Snowmass 2013 ➠ present

https://www.usqcd.org/documents/13flavor.pdf and [J. Butler et al, arXiv:1311.1076]

Quantity	CKM element	2013 expt. error	2007 forecast lattice error	2013 lattice error	2018 forecast lattice error	2021 FLAG Average
f_K/f_π	$\|V_{us}\|$	0.2%	0.5%	0.4%	0.15%	0.18 %
$f_+^{K\pi}(0)$	$\|V_{us}\|$	0.2%	–	0.4%	0.2%	0.18 %
f_D	$\|V_{cd}\|$	4.3%	5%	2%	$< 1\%$	0.3 %
f_{D_s}	$\|V_{cs}\|$	2.1%	5%	2%	$< 1\%$	0.2 %
$D \to \pi \ell \nu$	$\|V_{cd}\|$	2.6%	–	4.4%	2%	0.7 % [from 2212.12648]
$D \to K \ell \nu$	$\|V_{cs}\|$	1.1%	–	2.5%	1%	0.6 %
$B \to D^* \ell \nu$	$\|V_{cb}\|$	1.3%	–	1.8%	$< 1\%$	~1.5 % [from 2105.14019, 2304.03137, 2306.05657]
$B \to \pi \ell \nu$	$\|V_{ub}\|$	4.1%	–	8.7%	2%	~3 %
f_B	$\|V_{ub}\|$	9%	–	2.5%	$< 1\%$	0.7 % (0.6 % for f_{B_s})
ξ	$\|V_{ts}/V_{td}\|$	0.4%	2–4%	4%	$< 1\%$	1.3 %
Δm_s	$\|V_{ts}V_{tb}\|^2$	0.24%	7–12%	11%	5%	4.5 %
B_K	$\text{Im}(V_{td}^2)$	0.5%	3.5–6%	1.3%	$< 1\%$	1.3 %

QED corrections dominant source of theory error

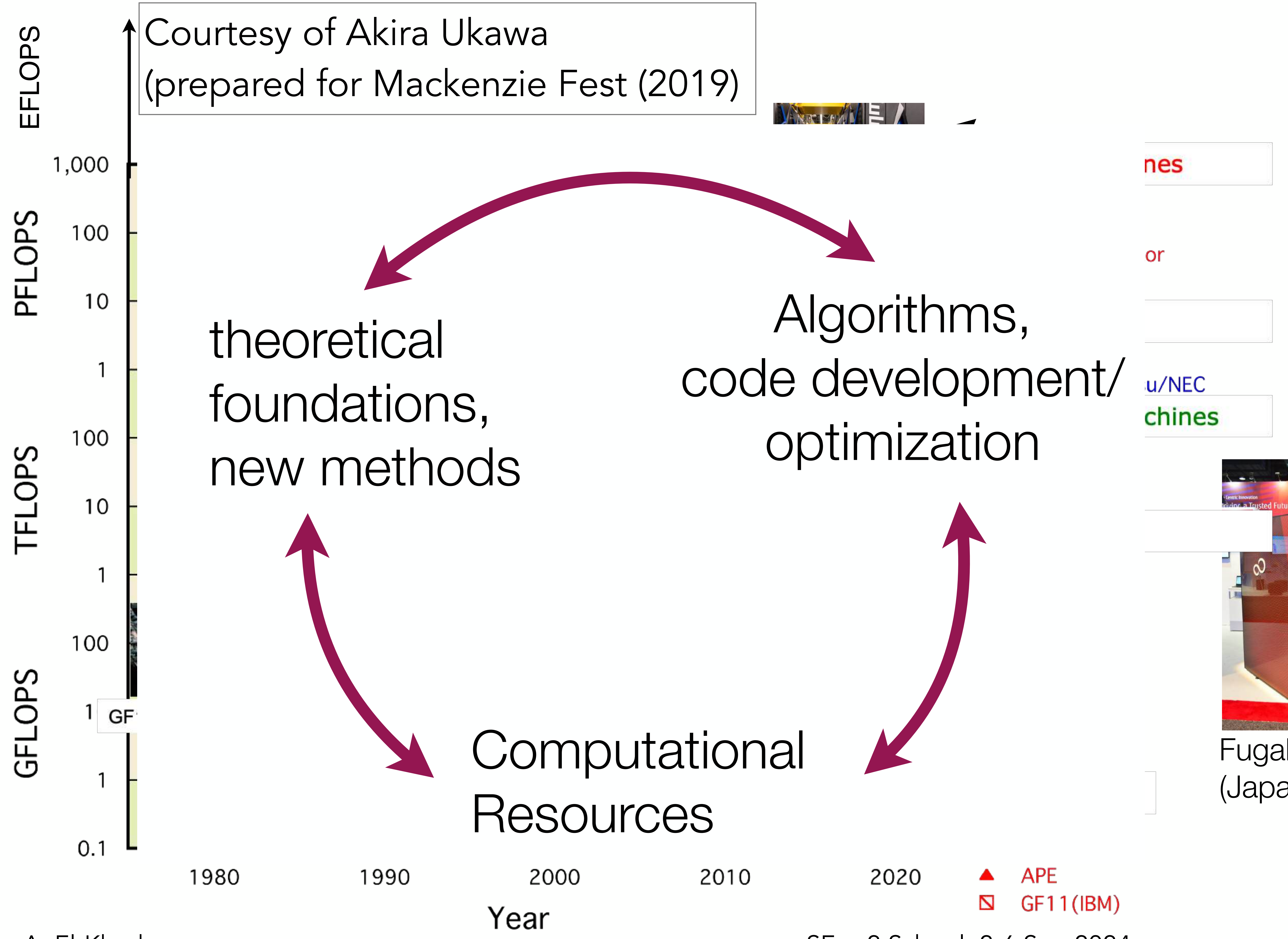
Timeline of computational resources



Fugaku 2020
(Japan)

Frontier 2022
(USA)

Timeline of computational resources



theoretical foundations, new methods

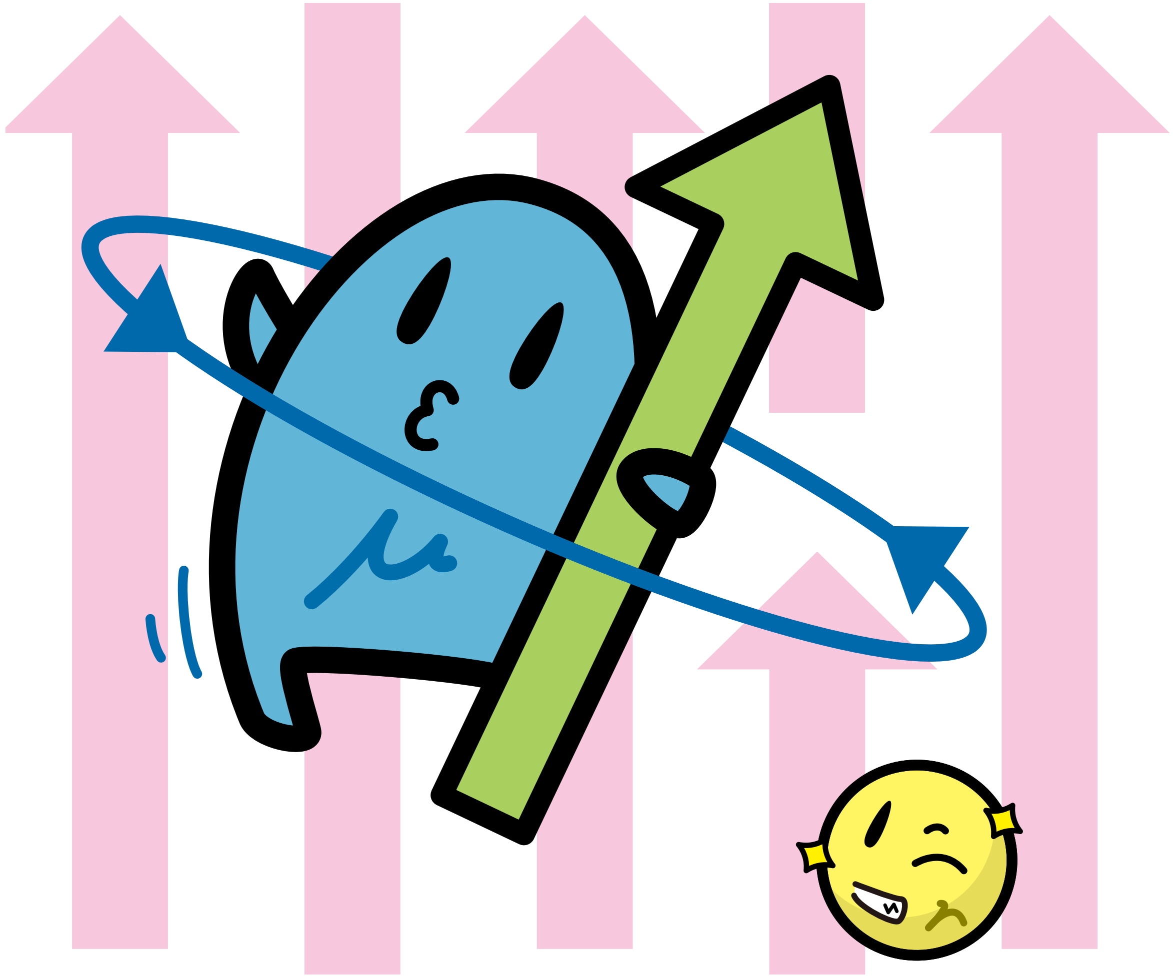
Algorithms, code development/ optimization

Computational Resources

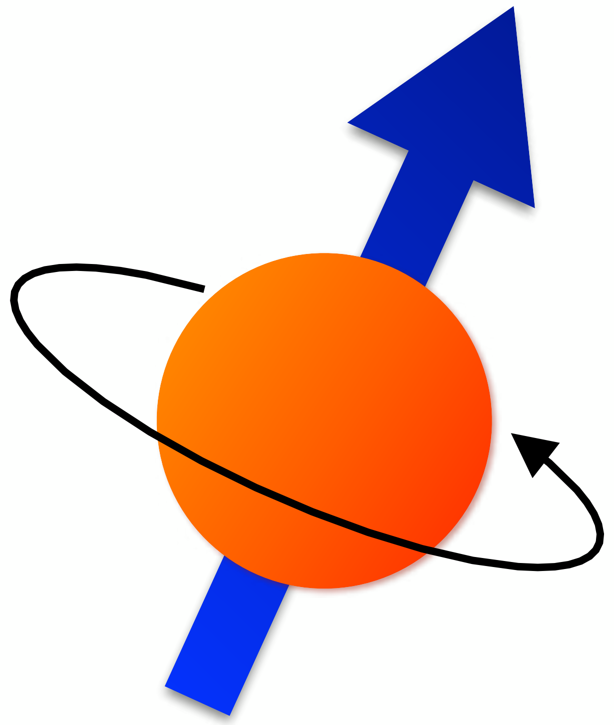
Fugaku 2020 (Japan)

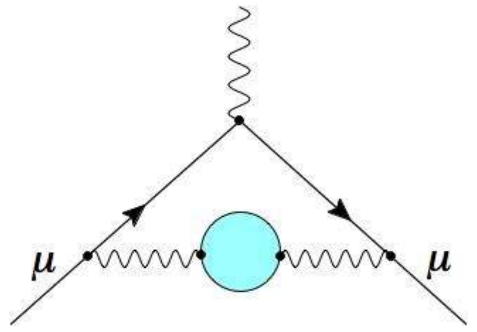
Frontier 2022 (USA)

Outline

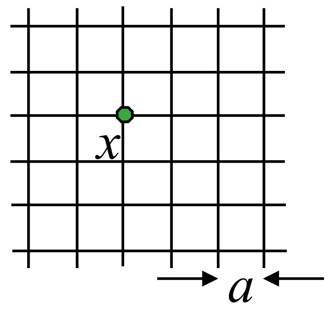


- Introduction
- Introduction to lattice QCD
  - sources of systematic errors
  - state of the art
- Lattice HVP
  - Introduction
  - systematic errors
  - separation prescription
  - Finite Volume (FV) effects
  - long-distance tail
  - windows in Euclidean time
  - Results and updates from Lattice 2024
- Summary and Outlook





Lattice HVP: Introduction



Leading order HVP contribution: $$a_\mu^{\mathrm{HVP,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int dq^2 \omega(q^2)\,\hat{\Pi}(q^2)$$

[B. Lautrup, A. Peterman, E. de Rafael, Phys. Rep 1972; E. de Rafael, Phys. Let. B 1994; T. Blum, PRL 2002]

- Calculate $a_\mu^{\mathrm{HVP,LO}}$ in Lattice QCD

  Start with correlation function of EM currents: $C(t) = \frac{1}{3}\sum_{i,x}\langle j_i^{\mathrm{EM}}(x,t) j_i^{\mathrm{EM}}(0,0)\rangle$ $\quad j_\mu^{\mathrm{EM}} = \sum_f q_f\,\bar{\psi}_f(x,t)\,\gamma_\mu\,\psi_f(x,t)$ $\quad f = u, d, s, c, \ldots$

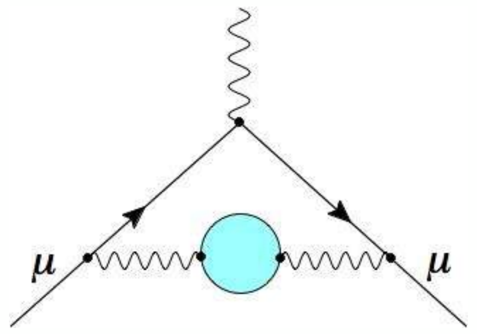
  Fourier transform yields $\hat{\Pi}(Q^2) = 4\pi^2 \int_0^\infty dt\, C(t)\left[t^2 - \frac{4}{Q^2}\sin^2\left(\frac{Qt}{2}\right)\right]$

  [D. Bernecker and H. Meyer, arXiv:1107.4388, EPJA 2011]

  so that $a_\mu^{\mathrm{HVP,LO}}$ can be obtained as an integral over Euclidean time, aka time momentum representation (TMR):

$$a_\mu^{\mathrm{HVP,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2\, w(Q^2)\,\hat{\Pi}(Q^2) = 4\,\alpha^2 \int_0^\infty dt\, C(t) \int_0^\infty dQ^2 w(Q^2)\left[t^2 - \frac{4}{Q^2}\sin^2\left(\frac{Qt}{2}\right)\right]$$

$$\Rightarrow \quad a_\mu^{\mathrm{HVP,LO}} = 4\,\alpha^2 \int_0^\infty dt\, C(t)\,\tilde{w}(t)$$

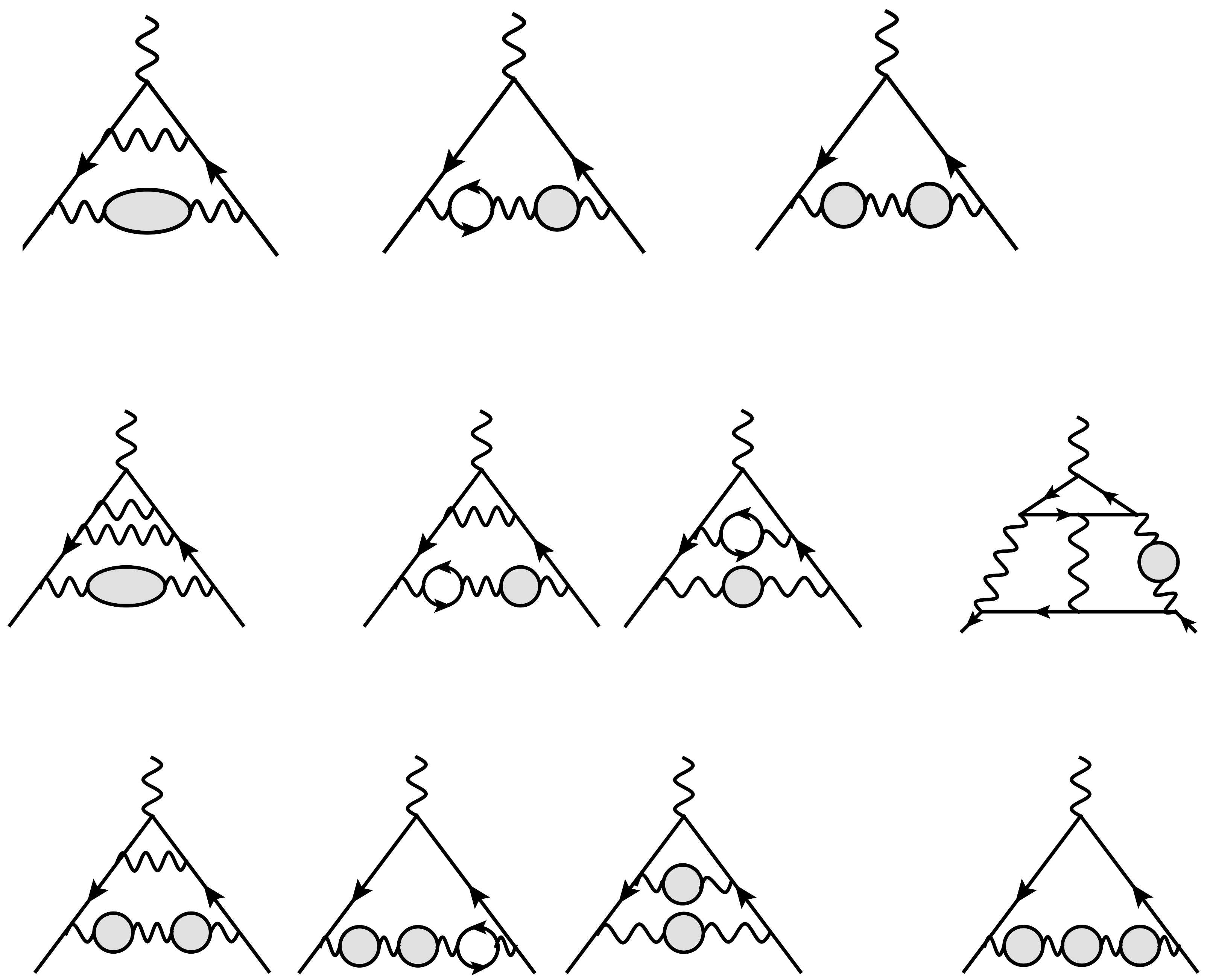


HVP: higher order (NLO, NNLO)

diagrams by T. Teubner

$$a_\mu^{\text{HVP,NLO}} = -9.83(7) \times 10^{-10}$$ [based on KNT 2019]

$$a_\mu^{\text{HVP,NNLO}} = 1.24(1) \times 10^{-10}$$ [Kurz et al, arXiv:1403.6400, PLB 2014]

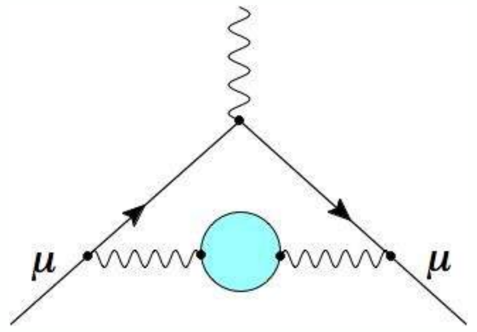


space-like NLO and NNLO HVP kernels for LQCD calculations and MUonE

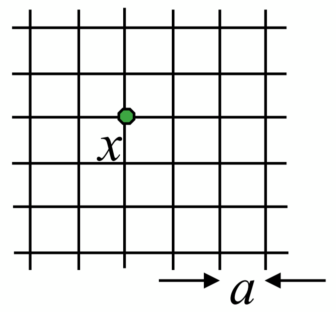
[Balsani et al, arXiv:2112.05704; Nesterenko, arXiv:2209.03217, arXiv: 2112.05009]

+ NLO HVP kernel in time-momentum representation (Euclidean time)

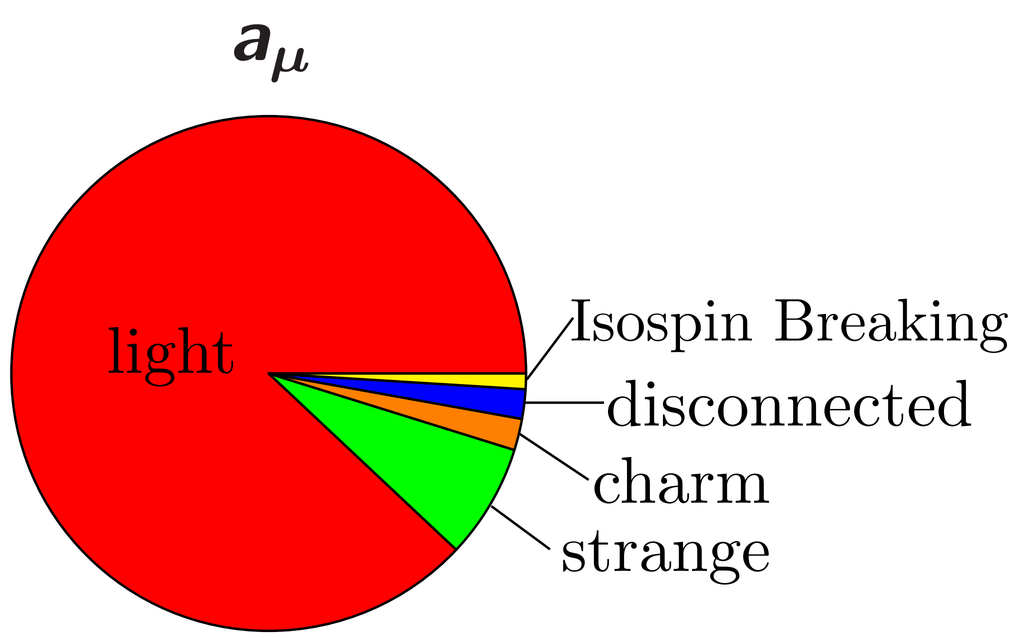
[Balsani et al, arXiv:2406.17940]



Lattice HVP: Introduction



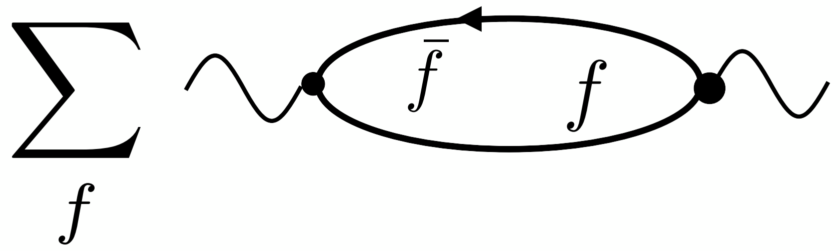
Calculate a_μ^{HVP} in Lattice QCD: $$a_\mu^{\mathrm{HVP,LO}} = \sum_f a_{\mu,f}^{\mathrm{HVP,LO}} + a_{\mu,\mathrm{disc}}^{\mathrm{HVP,LO}}$$



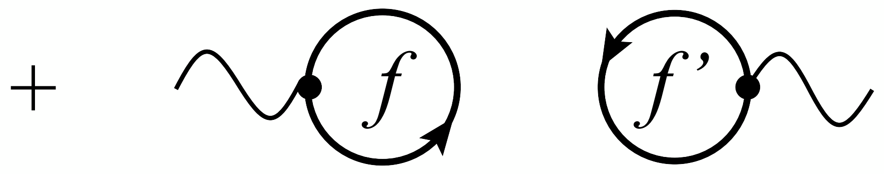
- Separate into connected for each quark flavor + disconnected contributions (gluon and sea-quark background not shown in diagrams)

  Note: almost always $m_u = m_d$

$f = ud, s, c, b$

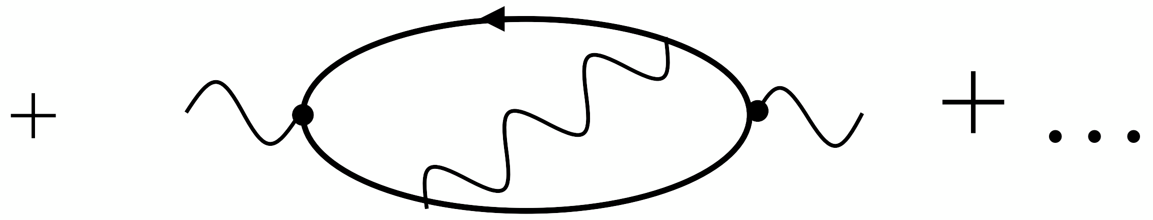


- light-quark connected contribution: $a_\mu^{\mathrm{HVP,LO}}(ud)$ ~90% of total
- s,c,b-quark contributions $a_\mu^{\mathrm{HVP,LO}}(s,c,b)$ ~8%, 2%, 0.05% of total



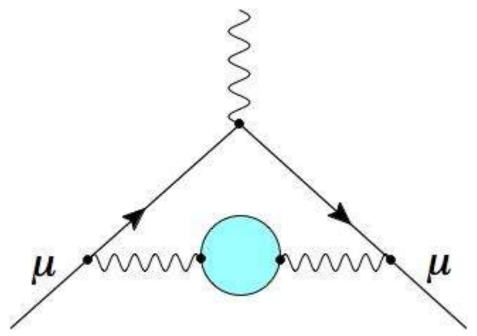
- disconnected contribution: $a_{\mu,\mathrm{disc}}^{\mathrm{HVP,LO}}$ ~2% of total

- need to add QED and strong isospin breaking ($\sim m_u - m_d$) corrections:

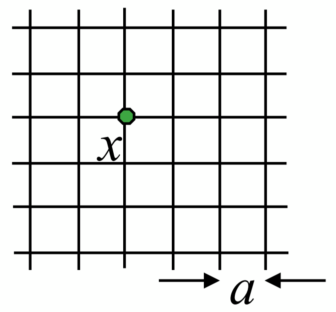


- Isospinbreaking (QED + $m_u \neq m_d$) corrections: $\delta a_\mu^{\mathrm{HVP,LO}}$ ~1% of total

$$a_\mu^{\mathrm{HVP,LO}} = a_\mu^{\mathrm{HVP,LO}}(ud) + a_\mu^{\mathrm{HVP,LO}}(s) + a_\mu^{\mathrm{HVP,LO}}(c) + a_{\mu,\mathrm{disc}}^{\mathrm{HVP,LO}} + \delta a_\mu^{\mathrm{HVP,LO}}$$

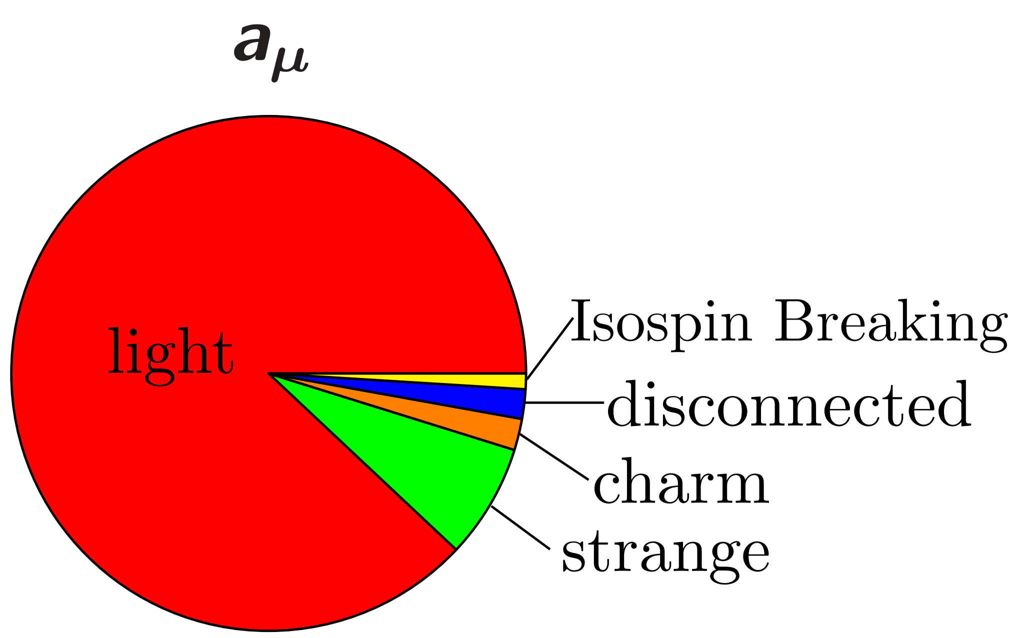


Lattice HVP: challenges



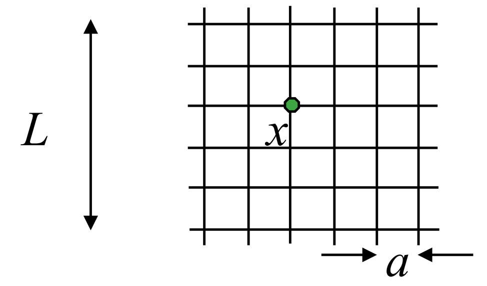
Calculate $a_\mu^{\rm HVP}$ in Lattice QCD: $$a_\mu^{\rm HVP,LO} = \sum_f a_{\mu,f}^{\rm HVP,LO} + a_{\mu,{\rm disc}}^{\rm HVP,LO}$$

$$a_\mu^{\rm HVP,LO} = a_\mu^{\rm HVP,LO}(ud) + a_\mu^{\rm HVP,LO}(s) + a_\mu^{\rm HVP,LO}(c) + a_{\mu,{\rm disc}}^{\rm HVP,LO} + \delta a_\mu^{\rm HVP,LO}$$



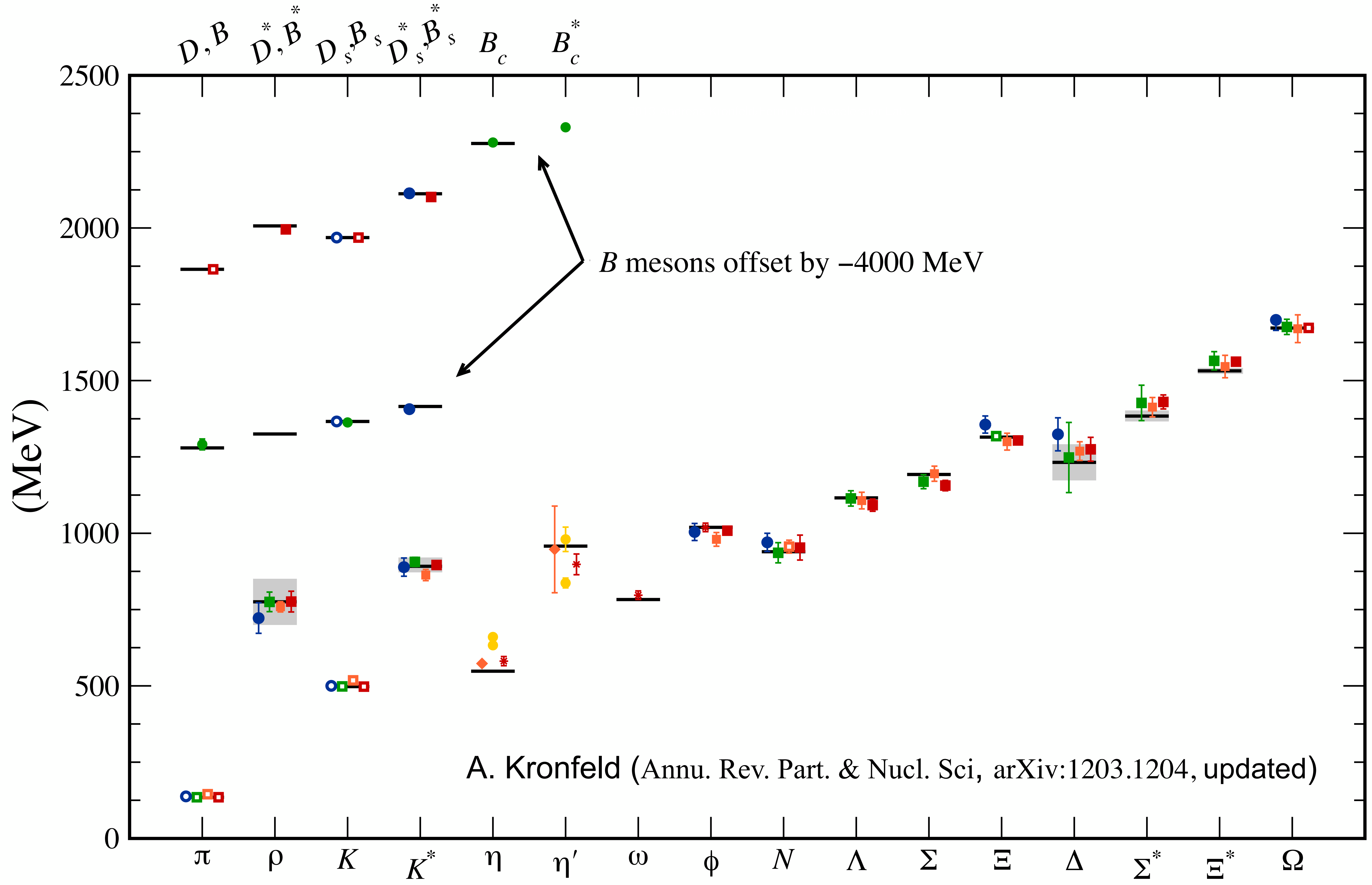
- $a_\mu^{\rm HVP,LO}$ needed with $< 0.5\,\%$ precision
- subpercent statistical precision:
  exponentially growing noise-to-signal in $C(t)$ as $t \to \infty$
  affects light-quark contributions
- sizable finite volume effects
- sensitivity to scale setting uncertainty
- control discretization effects
- quark-disconnected diagrams: control noise
- include isospin-breaking effects
  Separation of $a_\mu^{\rm HVP,LO}$ into $a_\mu^{\rm HVP,LO}(ud)$ and $\delta a_\mu^{\rm HVP,LO}$ is scheme dependent.

- light-quark connected contribution:
  $a_\mu^{\rm HVP,LO}(ud)$ ~90% of total
- *s*,*c*,*b*-quark contributions
  $a_\mu^{\rm HVP,LO}(s,c,b)$ ~8%, 2%, 0.05% of total
- disconnected contribution:
  $a_{\mu,{\rm disc}}^{\rm HVP,LO}$ ~2% of total
- Isospinbreaking (QED + $m_u \neq m_d$) corrections:
  $\delta a_\mu^{\rm HVP,LO}$ ~1% of total



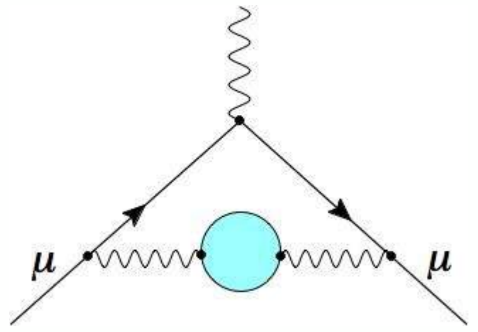
(lattice) QCD inputs

- free parameters in (lattice) QCD Lagrangian require experimental inputs
- **bare** quark masses, m_{ud}, m_s, m_c, m_b: fixed with exp. measured hadron masses, e.g., $M_\pi, M_K, M_{D_s}, M_{B_s}$
- lattice spacing in physical units (scale setting): f_π (or f_K M_Ω or …) ➠ α_s

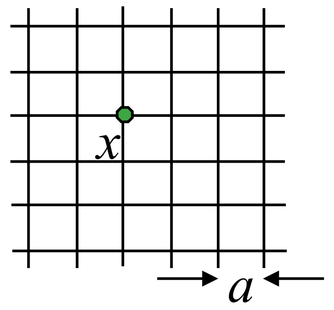


- all other quantities are pre/post dictions that can be compared to experiment.
- determinations of **renormalized** α_s from many different observables/methods: Wilson loops, current correlators, HQ potential, step scaling,…
- quark masses m_q: different intermediate renormalization schemes (nonperturbative or perturbative) before matching to $\overline{MS}$

➠ appendix



Lattice HVP: scale setting



- $a_\mu^{\rm HVP,LO}$ is dimensionless, but depends on dimensionful parameters (m_μ, m_q, etc…), which are expressed in terms of the lattice scale Λ:

$$a_\mu^{\rm HVP,LO} \equiv a_\mu^{\rm HVP,LO}\,(M_\mu, M_u, M_d, \ldots) \qquad M_\mu = \frac{m_\mu}{\Lambda}, \ldots$$

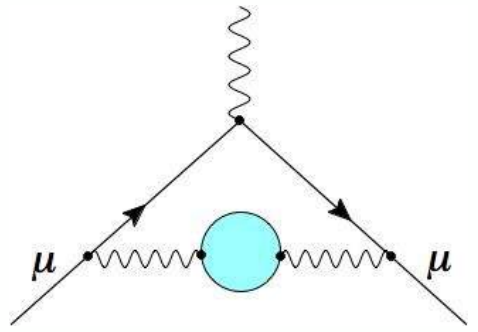
- uncertainty in $a_\mu^{\rm HVP,LO}$ due to the error in the scale determination, $\Delta\Lambda$:

[Della Morte et al, arXiv:1705.01775, JHEP 2017]

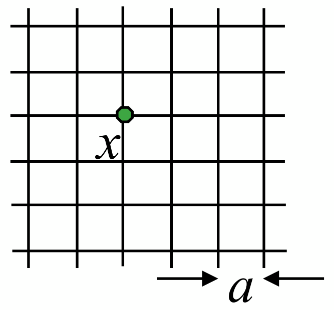
$$\Delta a_\mu^{\rm HVP,\ LO} = \left| \Lambda \frac{d a_\mu^{\rm HVP,\ LO}}{d\Lambda} \right| \times \frac{\Delta\Lambda}{\Lambda} = \left| M_\mu \frac{d a_\mu^{\rm HVP,\ LO}}{d M_\mu} \right| \times \frac{\Delta\Lambda}{\Lambda} \quad \Rightarrow \quad \frac{\Delta a^{\rm HVP,LO}}{a^{\rm HVP,LO}} \simeq 1.8 \frac{\Delta\Lambda}{\Lambda}$$

➠ need to determine lattice scale to high precision (< 0.2%)

- Scale setting quantities currently used in lattice QCD calculations:
  - f_π (or f_K) — depend on V_{ud} (V_{us})
    radiative QED corrections ~2%
  - Ω baryon mass — QED corrections are small (~0.1%)



Lattice HVP: separation prescription



A. Portelli @ Higgscentre workshop

- For an observable X one ideally wants an expansion (FLAG notation)

$$X^\phi = \bar{X} + X_\gamma + X_{\mathrm{SU}(2)}$$

  - $X_{\mathrm{SU}(2)}$: strong IB
  - X_γ: electromagnetic IB
  - $\bar{X}$: iso-symmetric

- A complete set of hadron masses defines X^ϕ **unambiguously**

- The separation in 3 contributions requires additional conditions, and are **scheme-dependent**

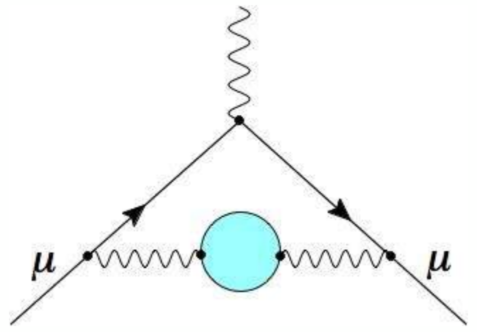
If different schemes are used by different groups, then comparisons must include scheme dependence

$$X_M(M,\alpha) = X^\phi + \frac{\partial X_M}{\partial M}(M - M^\phi) + (\alpha - \alpha^\phi)\frac{\partial X_M}{\partial \alpha}$$

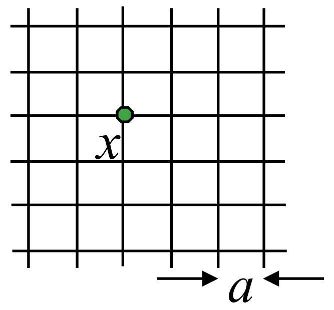
- **Prescription proposal** (both with $\alpha = 0$)

Pure QCD	*Iso-symmetric QCD*
$\hat{M}_{\pi^+} = 135.0\ \mathrm{MeV}$	$\bar{M}_\pi = 135.0\ \mathrm{MeV}$
$\hat{M}_{K^+} = 491.6\ \mathrm{MeV}$	$\bar{M}_K = 494.6\ \mathrm{MeV}$
$\hat{M}_{K^0} = 497.6\ \mathrm{MeV}$	

- Scale setting with $\hat{M}_{\Omega^-} = \bar{M}_{\Omega^-} = 1672.45\ \mathrm{MeV}$ possibly using a theory scale as proxy



Lattice HVP: separation prescription



- BMW 2024 [A. Boccaletti et al, arXiv:2407.10913] and [A. Risch @ Lattice 2024]

Decomposition of the neutral and charged kaon masses in three different schemes.

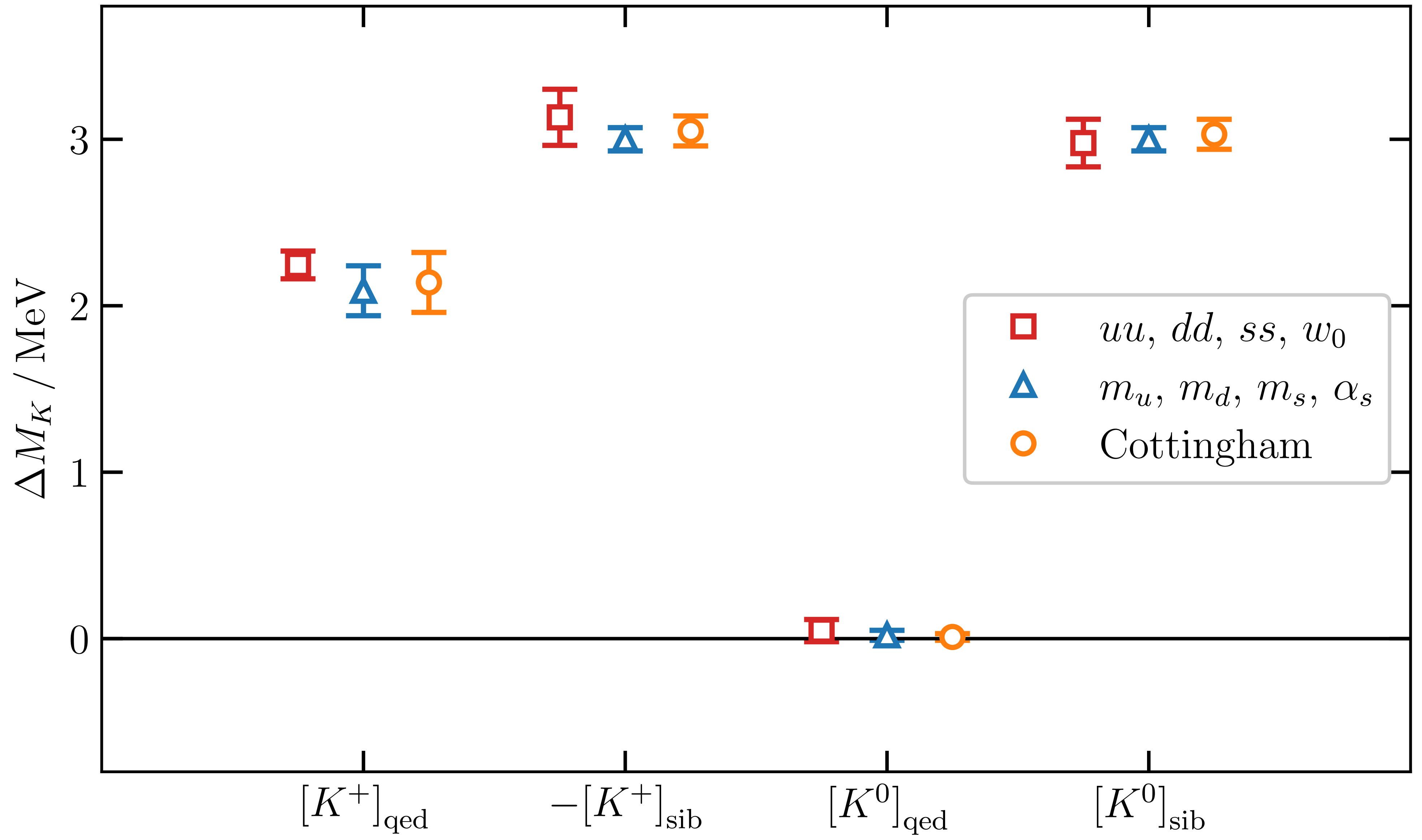
Decomposition in the $\{M_{\mathrm{uu}}, M_{\mathrm{dd}}, M_{\mathrm{ss}}, w_0\}$ scheme.

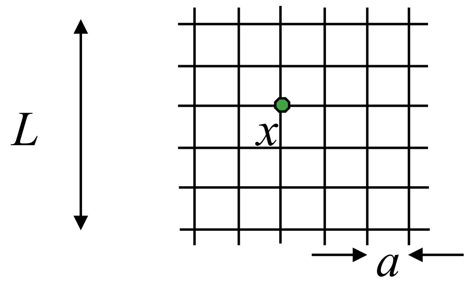
Agreement on decomposition of M_K indicates equivalence of schemes ($\Delta M_\pi \propto \alpha$ at LO):

- This work[9]: $\{M_{\mathrm{uu}}, M_{\mathrm{dd}}, M_{\mathrm{ss}}, w_0\}$
- Gasser-Rusetsky-Scimemi (GRS) scheme[10]: $\{m_{\mathrm{u}}, m_{\mathrm{d}}, m_{\mathrm{s}}, \alpha_{\mathrm{s}}\}$ in $\overline{\mathrm{MS}}$ at $\mu = 2\,\mathrm{GeV}$
- Cottingham-formula based decomposition[11]: Relates electromagnetic self-energy to forward Compton tensor

$\Rightarrow$ Good agreement with GRS and Cottingham-formula based schemes

[9] Borsanyi 2021.
[10] Di Carlo et al. 2019.
[11] Stamen et al. 2022.





finite volume (FV) effects

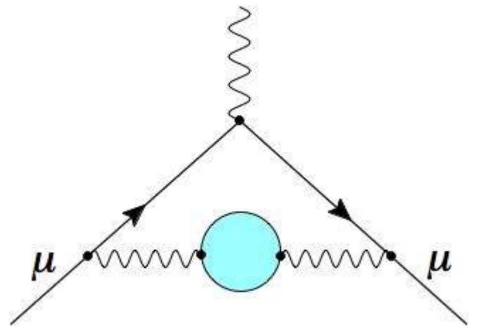
QCD only, with one stable hadron in initial/final state:

FV EFT (Lüscher): If L large enough, so that $m_\pi L \gtrsim 4$ ➠ leading FV error $\sim e^{-m_\pi L}$

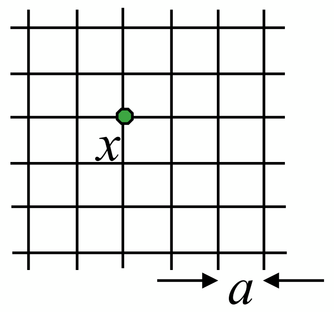
➠ residual FV effects can usually be quantified in ChPT

Or: direct FV study, comparing results at several different finite volumes (with other parameters fixed)

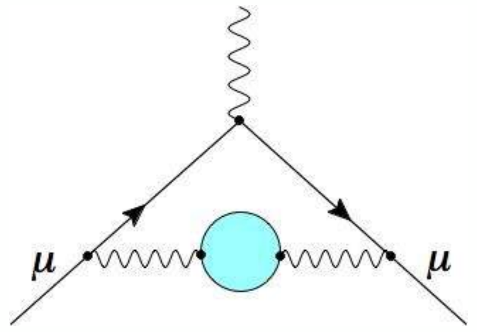
errors on the FV corrections depend on the statistical precision the FV study



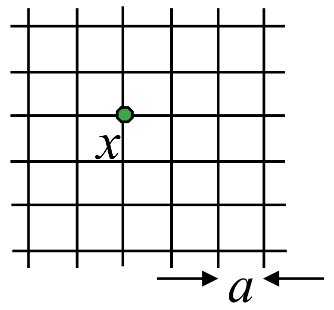
Lattice HVP: finite volume (FV) effects



- ☆ expected size (based on ChPT) ~ 3% on typical lattice volumes with $M_\pi L \gtrsim 4$
  - ➠ large volume studies are needed to quantify FV effects for sub-percent precision
- ☆ EFT and EFT-inspired approaches:
  - NNLO ChPT [Bijnens + Relefors, arXiv:1710.04479; Aubin et al, 2019,2020, 2022,…]
  - relativistic pion EFT [Hansen & Patella, arXiv:2004.03935]
  - spectral representation using $F_\pi(k)$ + Gounaris-Sakurai parameterization (MLLGS)
    [Meyer 2011; Francis et al 2013; Lellouch & Lüscher, 2001]
    use Lüscher and Lellouch-Lüscher equations to obtain FV E_n, A_n and compute $C(t,\infty) - C(t,L)$
  - Chiral Model: ChPT + ρ [Jegerlehner & Szafron, arXiv:1101.2872; HPQCD, arXiv:1601.03071]
  - For staggered fermions: modify FV EFTs to account for splittings between pions with different ``tastes"
- ☆ large-volume studies by BMW (2020), RBC/UKQCD, PACS



Lattice HVP: finite volume (FV) effects



Chiral Perturbation Theory (ChPT)

- Compute $a_\mu^{\rm HVP}$ in ChPT in both IV and FV, take the difference (note, $p_n = 2\pi\, n/L$):

$$\int \frac{dp}{2\pi} F(p) \quad \rightarrow \quad \frac{1}{L}\sum_{p_n} F(p_n) \qquad\qquad \Delta a_\mu^{\rm HVP} = a_\mu^{\rm HVP}(\infty) - a_\mu^{\rm HVP}(L)$$

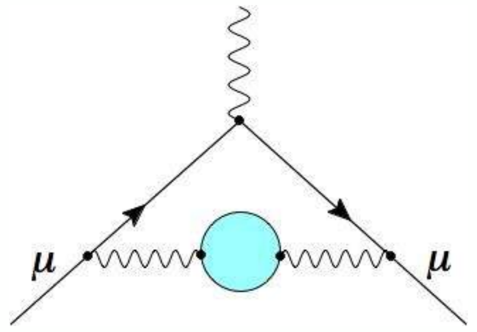
- can obtain $a_\mu^{\rm HVP}$ either from computing

  $\Pi_{\mu\nu}(q)$ [Bijnens + Relefors, arXiv:1710.04479] or $C(t,L)$ [Aubin et al, 2019, 2020, 2022]

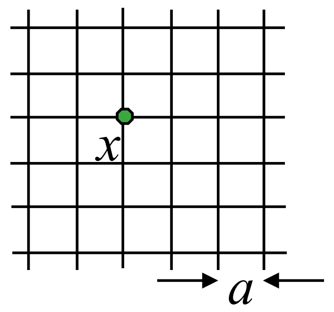
- for example, in NLO ChPT [Aubin et al, arXiv:1905.09307]: $C^{\rm NLO}(t,L) = \frac{10}{9}\frac{1}{3}\frac{1}{L^d}\sum_{\vec p}\frac{\vec p^2}{E_p^2}\, e^{-2E_p t}$
  which yields

$$\Delta a_\mu^{\rm HVP,\,NLO} = \frac{10}{9}\frac{\alpha^2}{6\pi^2}\sum_{n^2=1}^{\infty}\frac{Z_{00}(0,n^2)}{nL}\int_0^\infty dp\,\frac{p^3}{E_p^2}\,\sin(npL)F(p^2)$$

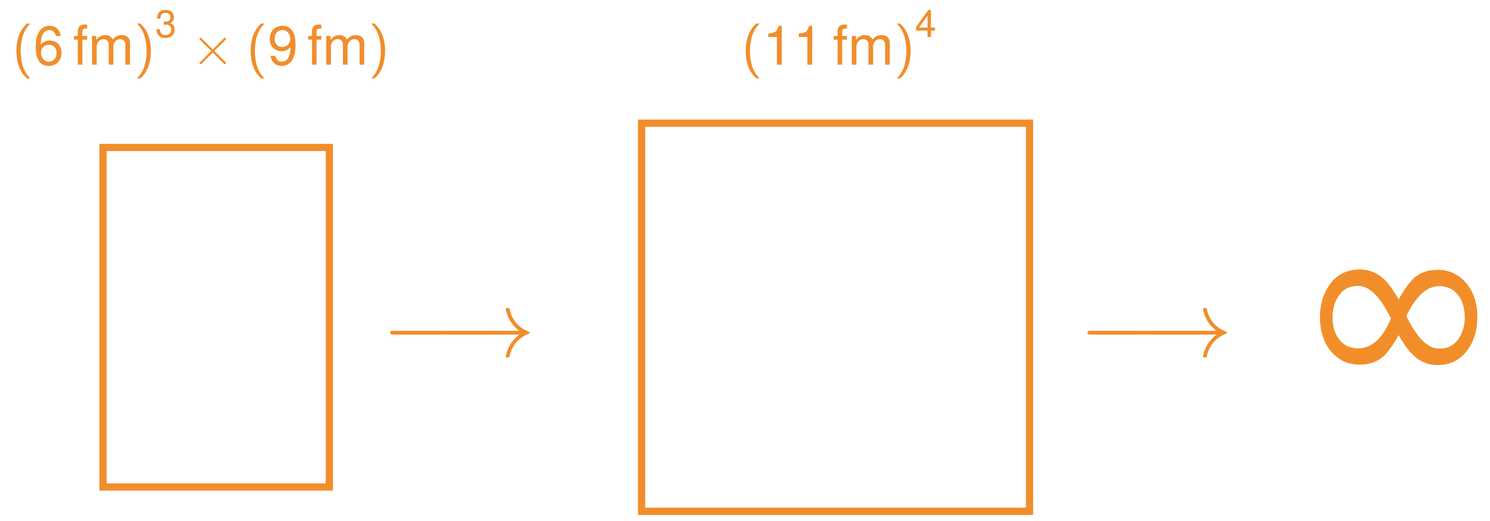
where $F(p^2)$, $Z_{00}(0,n^2)$ are given in arXiv:1905.09307.



Lattice HVP: finite volume effects



Finn Stokes (BMWc) @ Lattice 2021



4HEX	18.1 ± 2.4		$\times 10^{-10}$
NNLO χPT	15.7	0.6 ± 0.3	$\times 10^{-10}$
MLLGS	17.8		$\times 10^{-10}$

Model comparison

- Two more models for finite L (but not T)
  - Generic field-theory approach [Hansen & Patella '19, '20] (HP) relates the finite-size effect to $F_\pi(k)$
  - Rho-pion-gamma model [Chakraborty et al '17] (RHO) incorporates the $\rho(770)$ resonance directly into a χPT-like framework

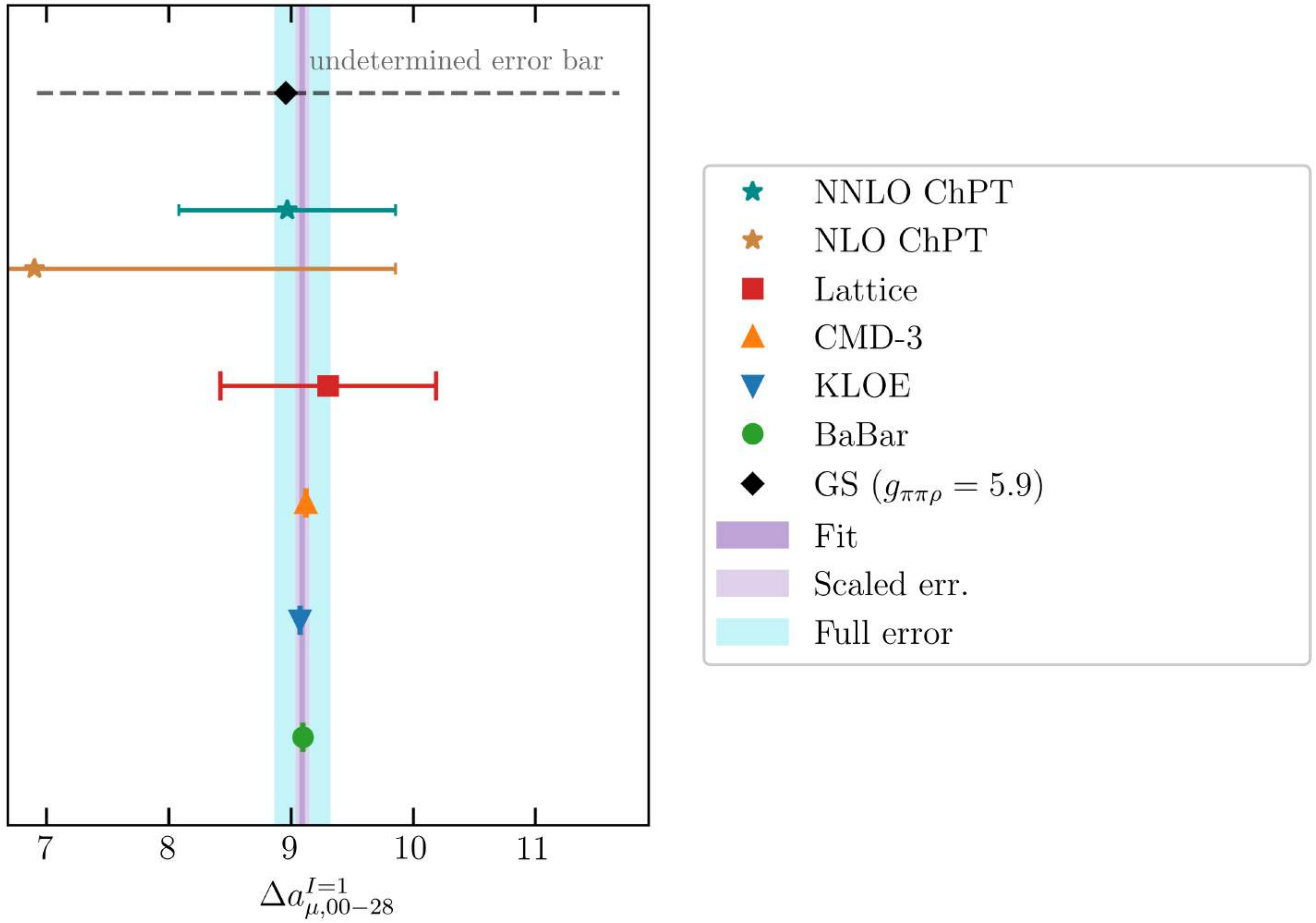
- Compare finite L corrections for reference volume in infinite-T limit
- All four models agree within $\sim 2.5 \times 10^{-10}$

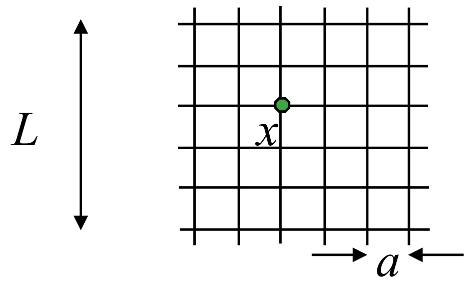
NNLO χPT	16.7	$\times 10^{-10}$
MLLGS	18.8	$\times 10^{-10}$
HP	17.7	$\times 10^{-10}$
RHO	16.2	$\times 10^{-10}$

- Other direct calculations by RBC/UKQCD [Lehner @ 2019 INT workshop], and Shintani & Kuramashi [PRD 2019] are consistent (but with larger errors).

[A. Lupo for BMW+DMZ @ Lattice 2024]

- Compare FV correction from BMW20 (4HEX) with HP and MLL using a data-driven evaluations of the parameters.





Lattice QCD Introduction

finite volume (FV) effects

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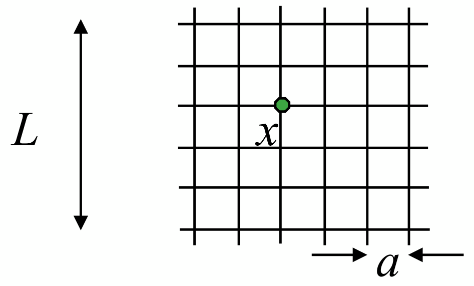
Or: direct FV study, comparing results at several different finite volumes (with other parameters fixed)

errors on the FV corrections depend on the statistical precision the FV study

➠ H. Meyer lecture



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FV EFT (Luescher): If L large enough, so that $m_\pi L \gtrsim 4$ ➠ leading FV error $\sim e^{-m_\pi L}$

➠ residual FV errors can usually be quantified in ChPT

Or: direct FV study, comparing results at several different finite volumes (with other parameters fixed)

errors on the FV corrections depend on the statistical precision the FV study

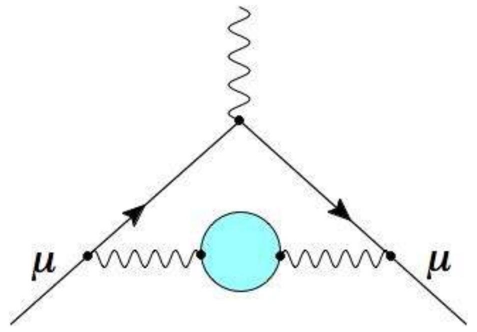
QCD + QED$_X$:

need a scheme to treat massless photon in FV: ➠ FV effects $\sim 1/L^n$

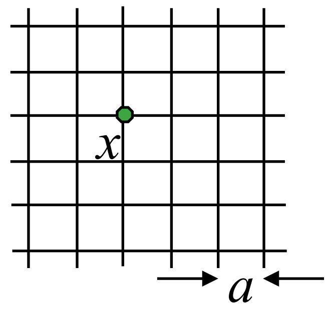
- QED$_L$ — $\sum_{\vec{x}} A_\mu(\vec{x}, t) = 0;$ ….
- QED$_m$ ($m_\gamma > 0$) take infinite volume limit before $m_\gamma \to 0$
- QED$_\infty$ (compute n-point correlation functions in LQCD + inf. volume QED)

➠ H. Meyer lecture

➠ see talks at MITP workshop on isospin breaking corrections



Lattice HVP: long-distance tail



$$C(t) = \frac{1}{3}\sum_{i,x}\langle j_i^{\rm EM}(x,t)\, j_i^{\rm EM}(0,0)\rangle$$

[M. Lynch @ Lattice 2024]
(FNAL/HPQCD/MILC)

- Use improved statistical estimators
  [BMW, RBC/UKQCD, Aubin et al, …]

Low mode improvement:

$$C^\Gamma_{\rm RW}(t) = \sum_i {\rm Tr}\Big\{S_i^{\rm RW}\Gamma\gamma^5 S_i^{\rm RW\dagger}\gamma^5\Gamma\Big\}, \quad S_i^{\rm RW}(x) = M^{-1}(x;y)\eta_i(y)$$

Low-mode propagator

$$M_L^{-1} = \sum_n^{N_e} \frac{1}{\lambda_n} v_n v_n^\dagger,$$

$$\lambda_n = \begin{cases} m + i\tilde\lambda_n & (n \bmod 2 = 0) \\ m - i\tilde\lambda_n & (n \bmod 2 = 1)\end{cases}$$

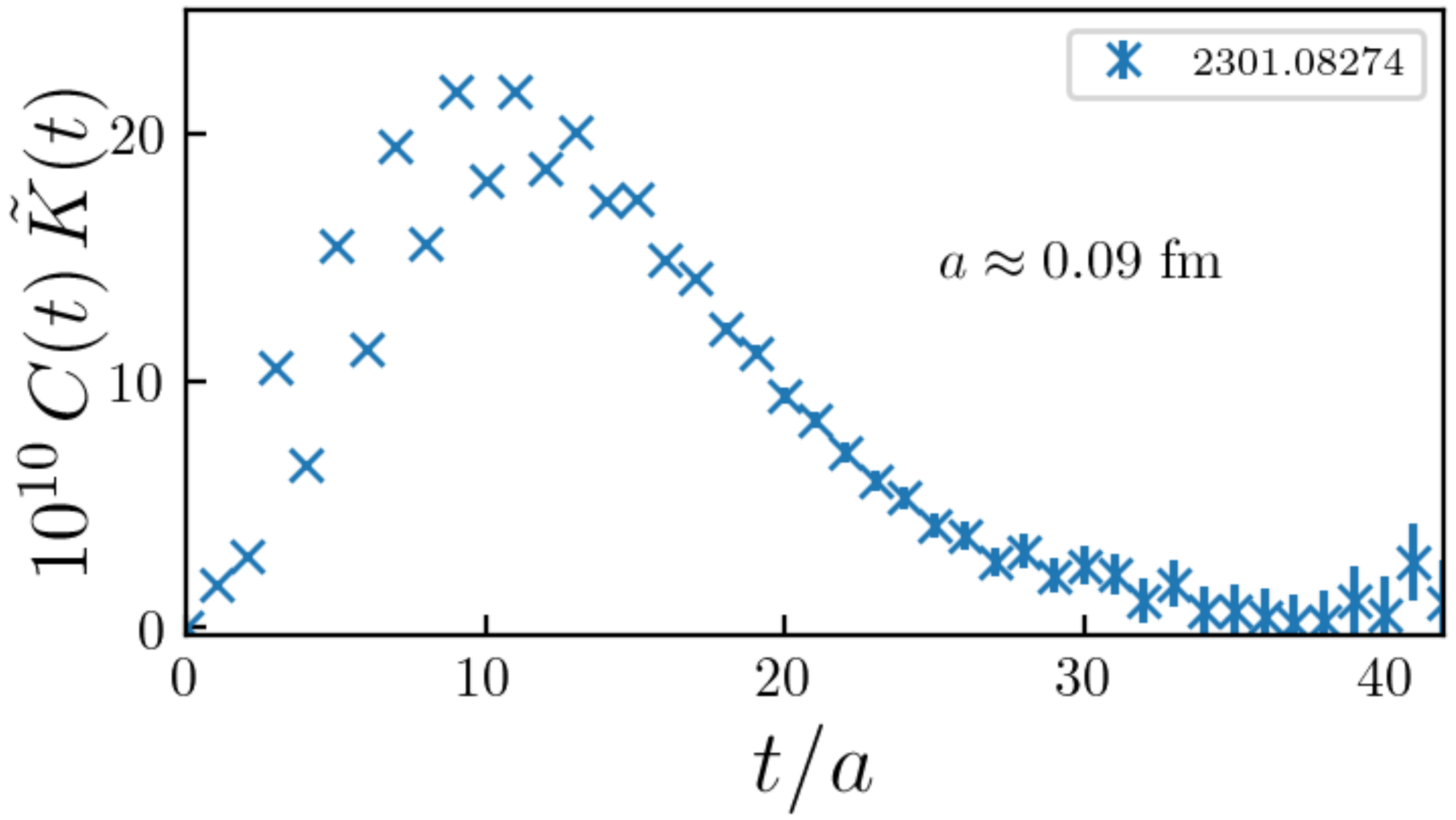
Low-mode–improved random wall estimator

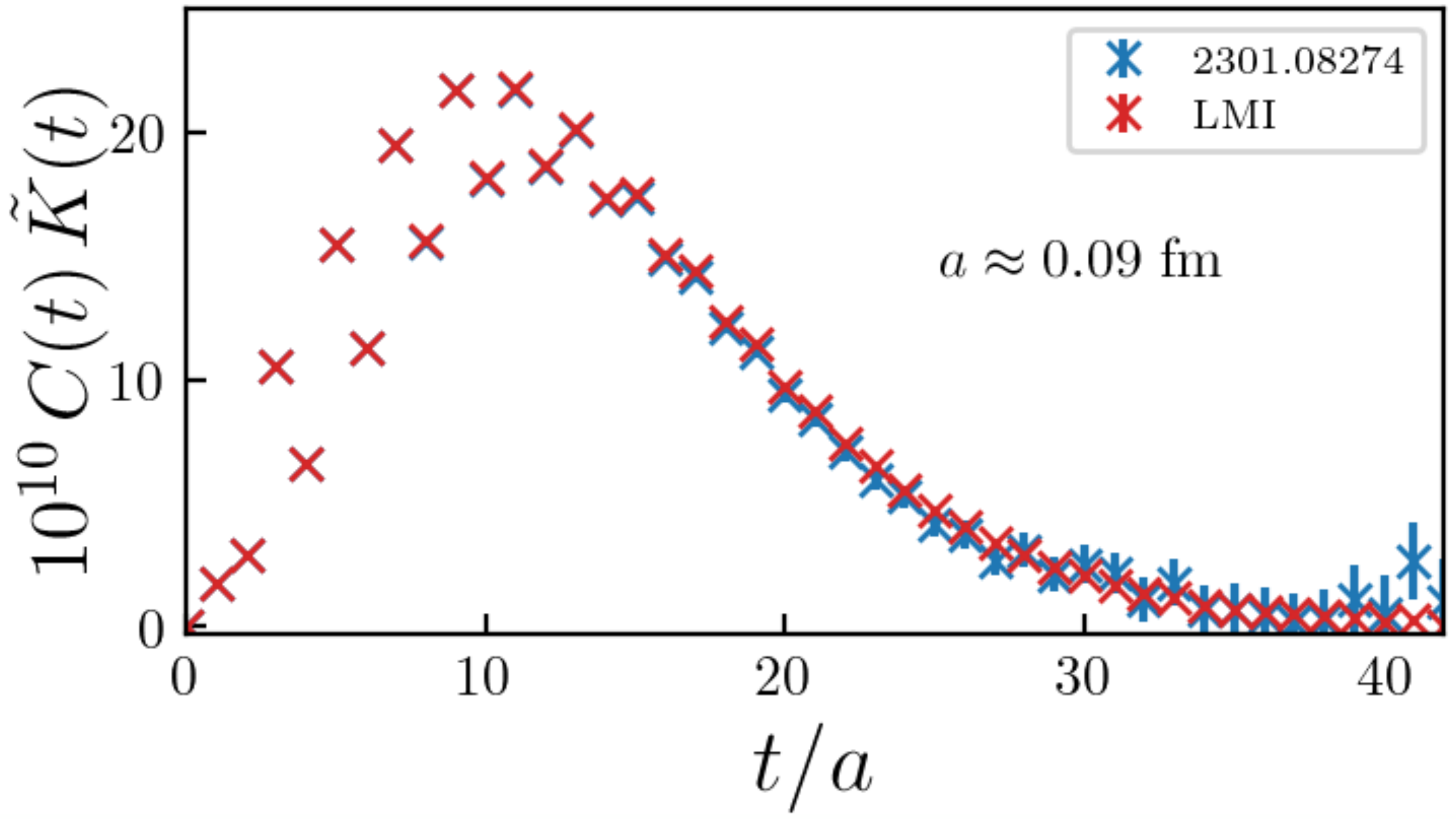
$$C^\Gamma_{\rm LMI}(t) = C^\Gamma_{\rm RW}(t) - C^\Gamma_{\rm RW,LL}(t) + C^\Gamma_{\rm LL}(t)$$

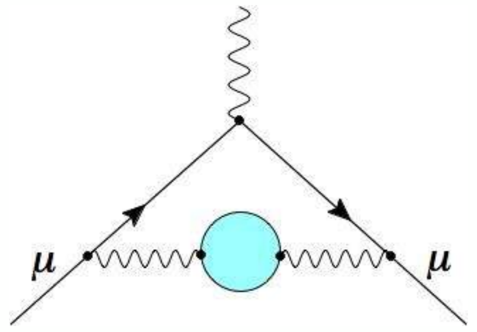
Exact low-mode contribution

$$C^\Gamma_{\rm LL}(t) = \sum_i {\rm Tr}\{M_L^{-1}\Gamma M_L^{-1}\Gamma\}$$

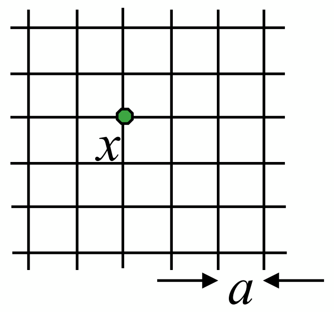
Big improvement in statistical errors at large Euclidean times.







Lattice HVP: long-distance tail



$$C(t) = \frac{1}{3}\sum_{i,x}\langle j_i^{\mathrm{EM}}(x,t)\, j_i^{\mathrm{EM}}(0,0)\rangle$$

- Start with spectral decomposition: $C(t) = \sum_{n=0}^{\infty} |A_n|^2 e^{-E_n t}$
  - ✦ bounding method:

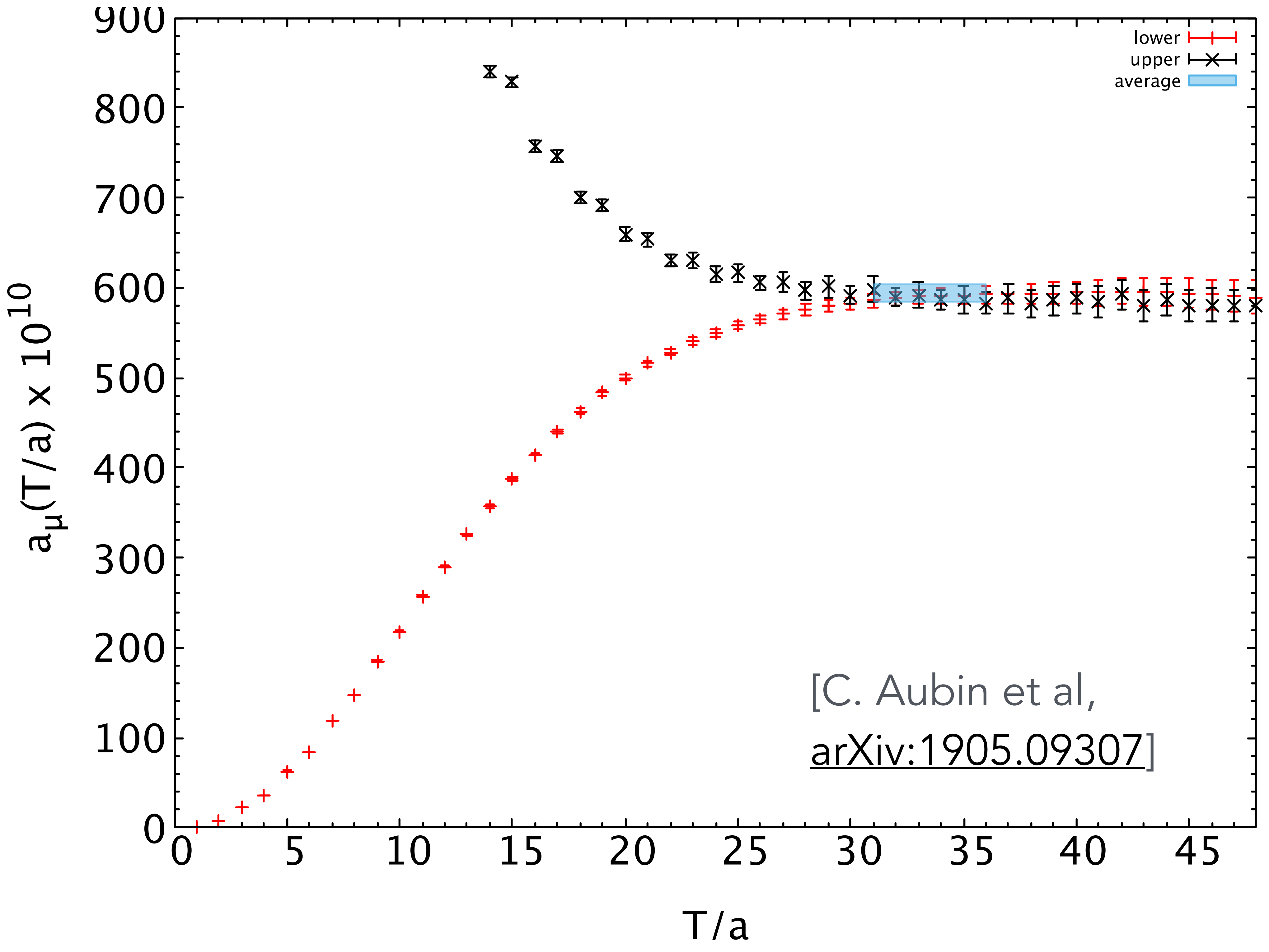
[Borsanyi et al, PRL 2018, Blum et al, PRL 2018]

for $t > t_c$: $0 \le C(t_c)\, e^{-\bar{E}_{t_c}(t-t_c)} \le C(t) \le C(t_c)\, e^{-E_0(t-t_c)}$

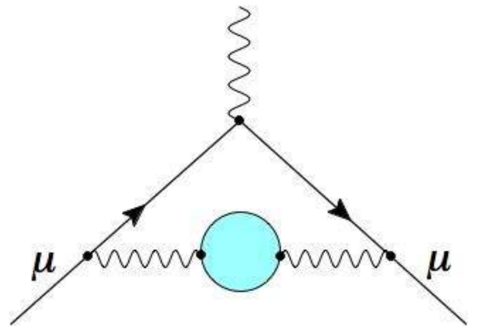
$\bar{E}_{t_c}$: effective mass of $C(t)$ at t_c

E_0: ground state energy

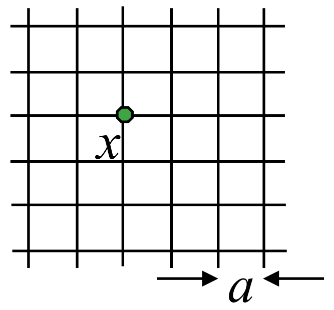
replace $G(t > t_c)$ with upper and lower bound, vary t_c

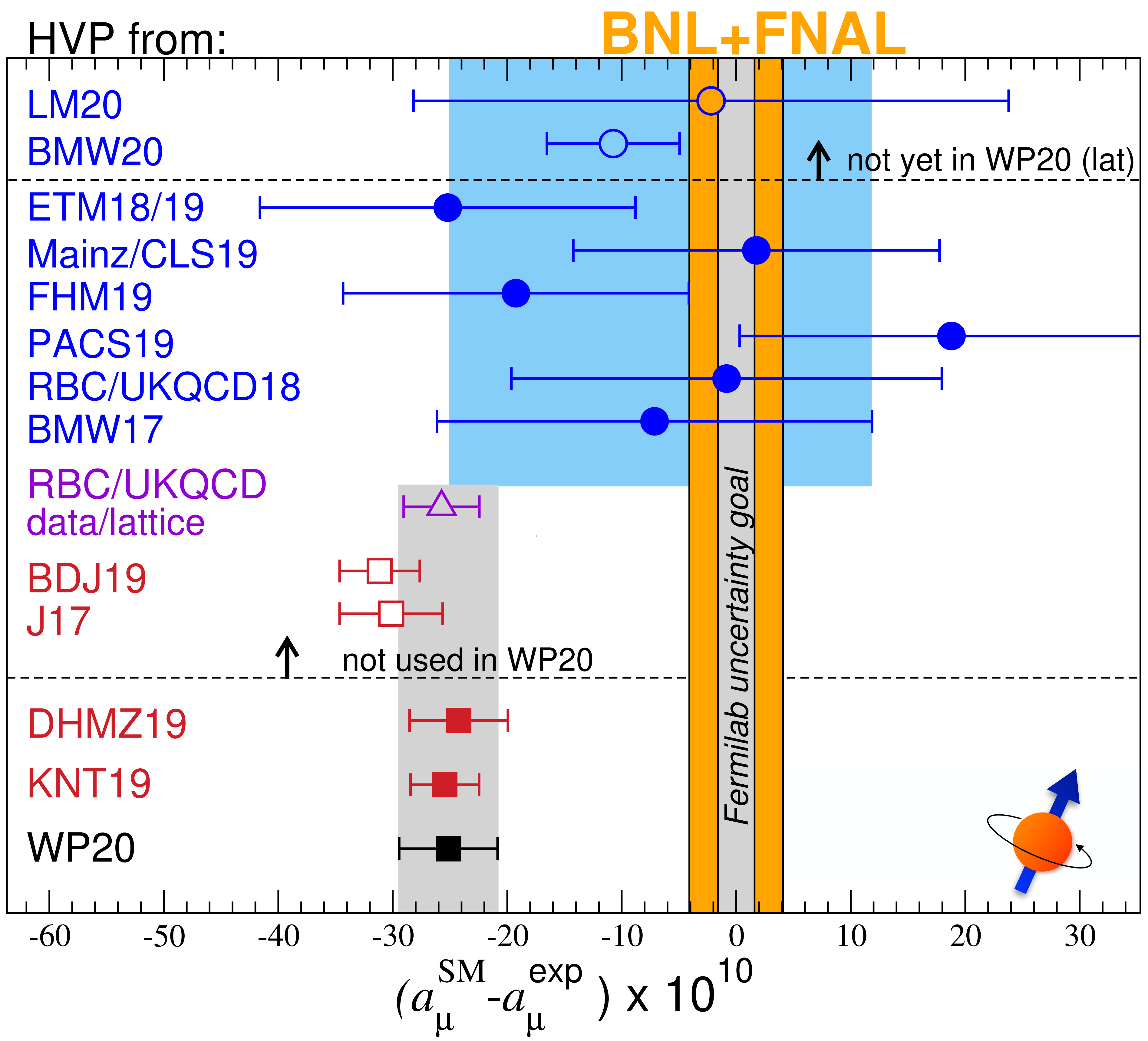


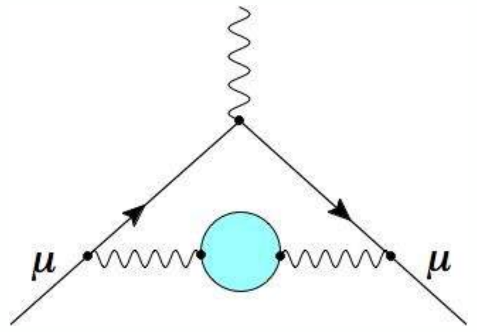
[C. Aubin et al, arXiv:1905.09307]



Lattice HVP: results







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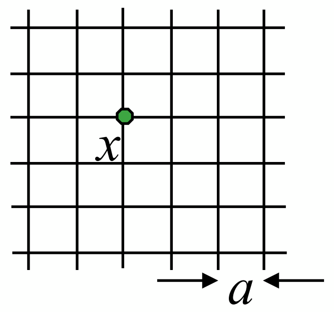
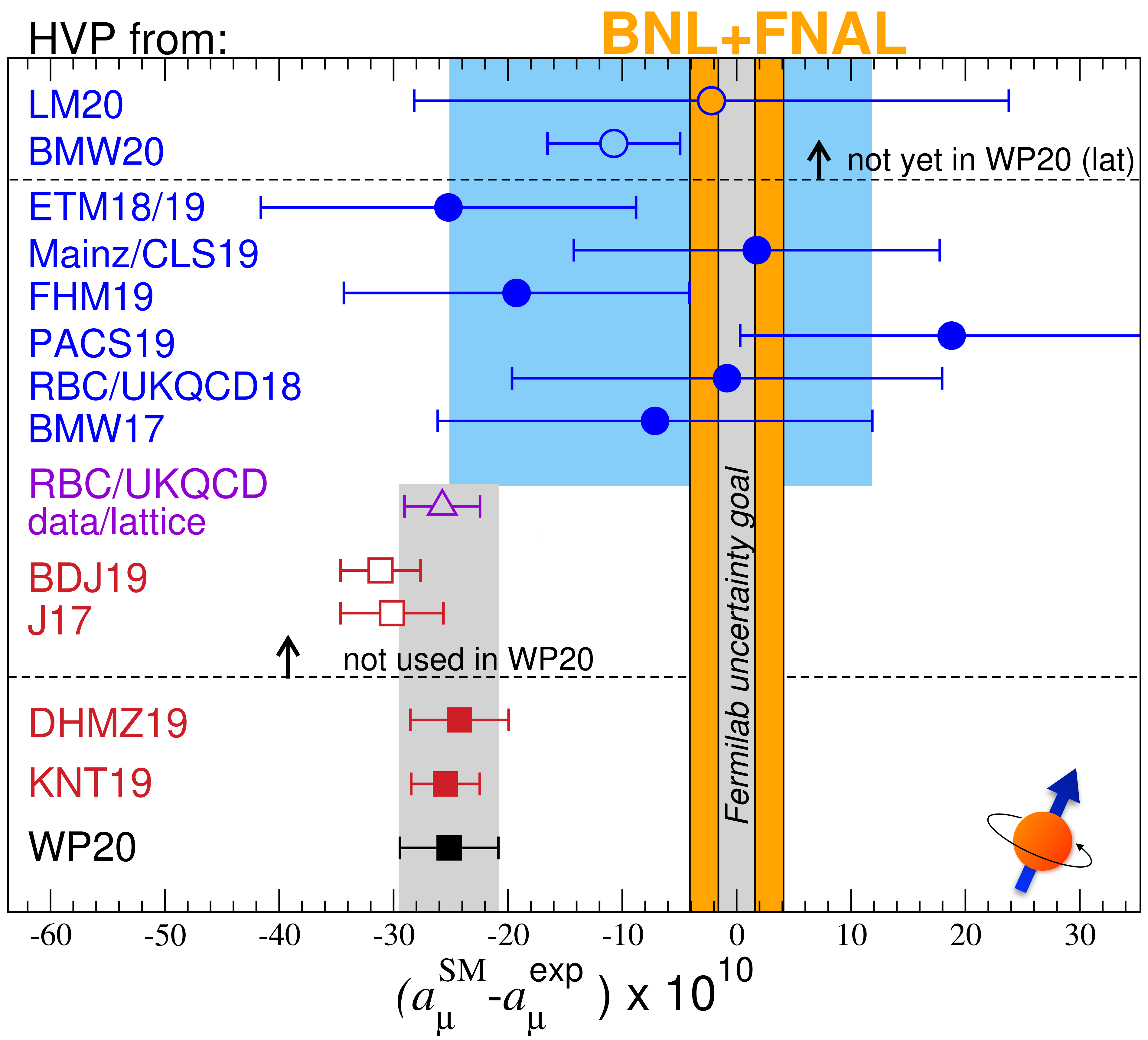
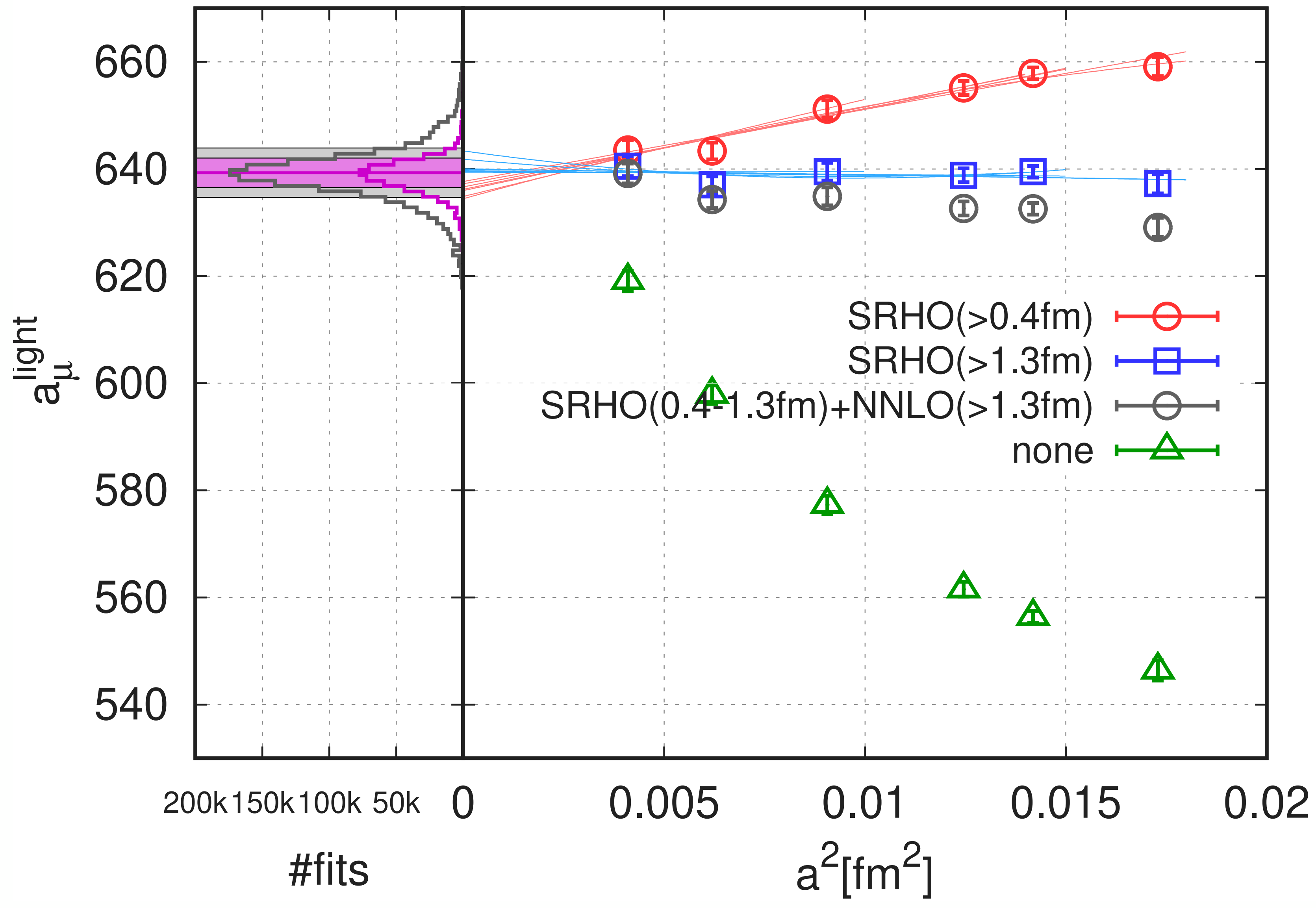


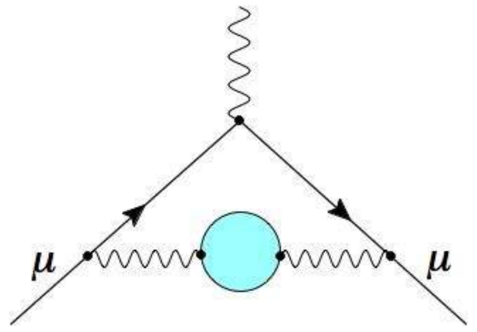


BMW 20 [Sz. Borsanyi et al, arXiv:2002.12347, 2021 Nature]

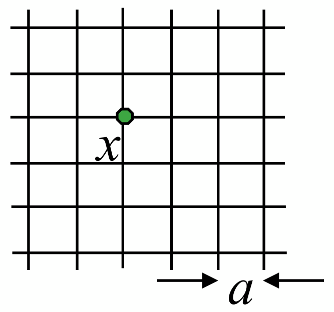
first complete LQCD calculation with sub-percent precision (0.8 %)



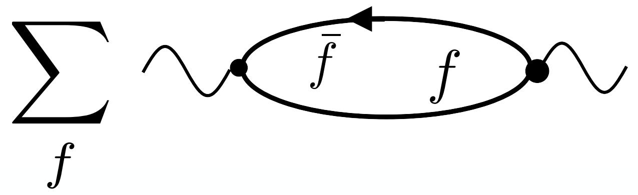
- total error in BMW calculation is dominated by light-quark connected contribution.
- uncertainty dominated by systematic errors; use model averaging (MA)
- Large taste-breaking effects with BMW set-up affect the continuum extrapolation: uncorrected data not easily fit to power series



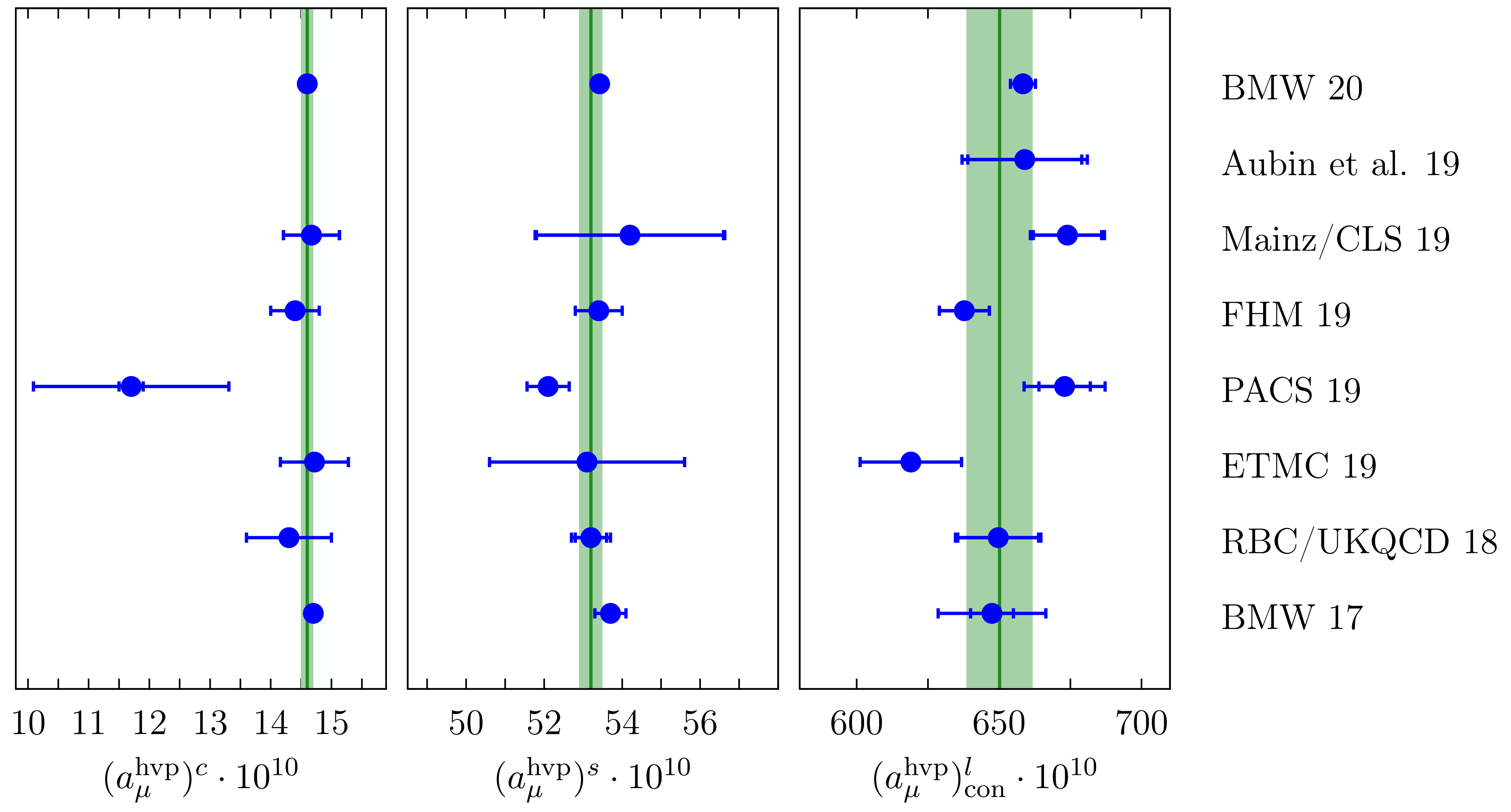
Lattice HVP: summary of contributions

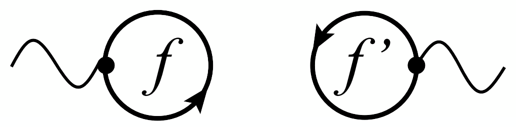


H. Wittig @ Lattice HVP workshop

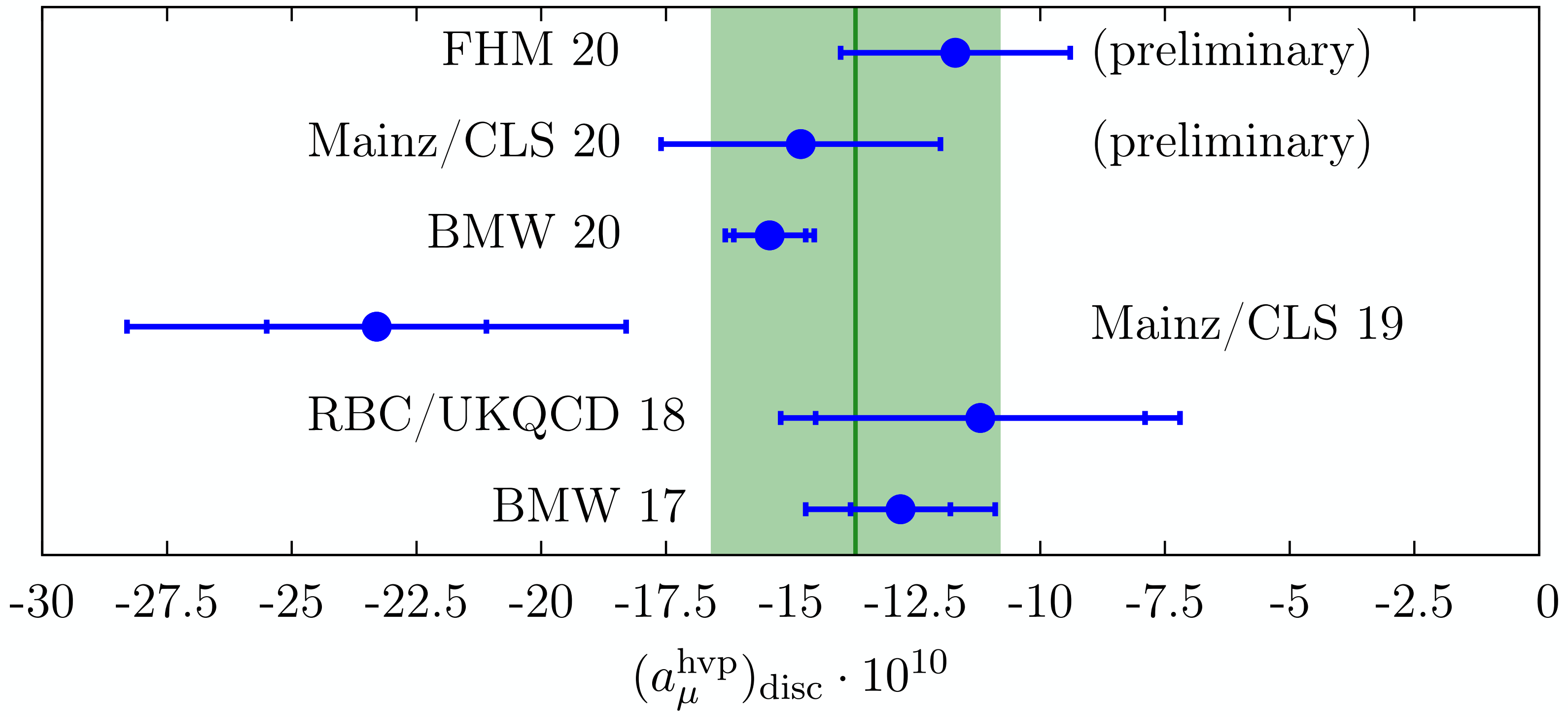


- Charm, strange contributions already well determined.

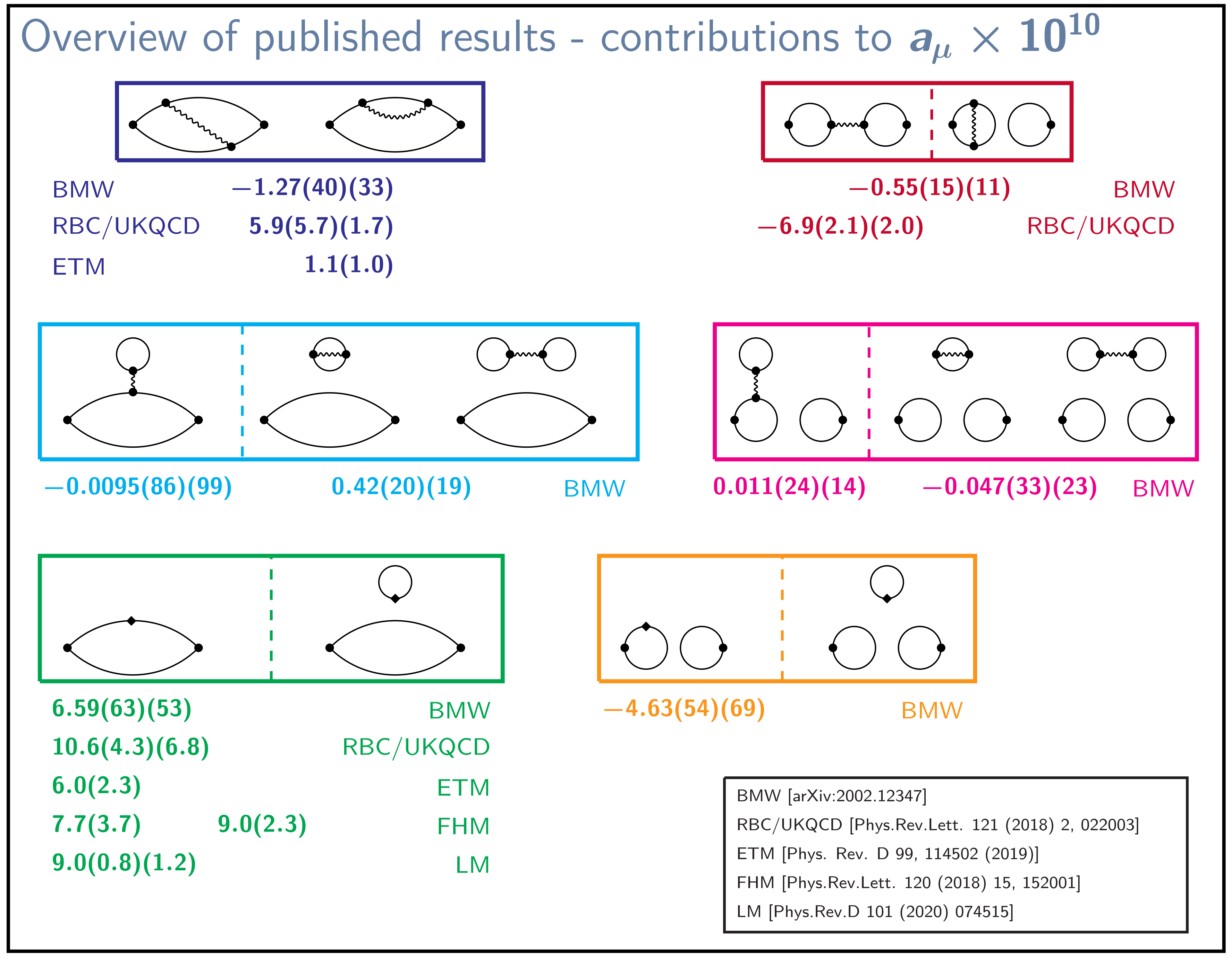




Consistent results with increasing precision

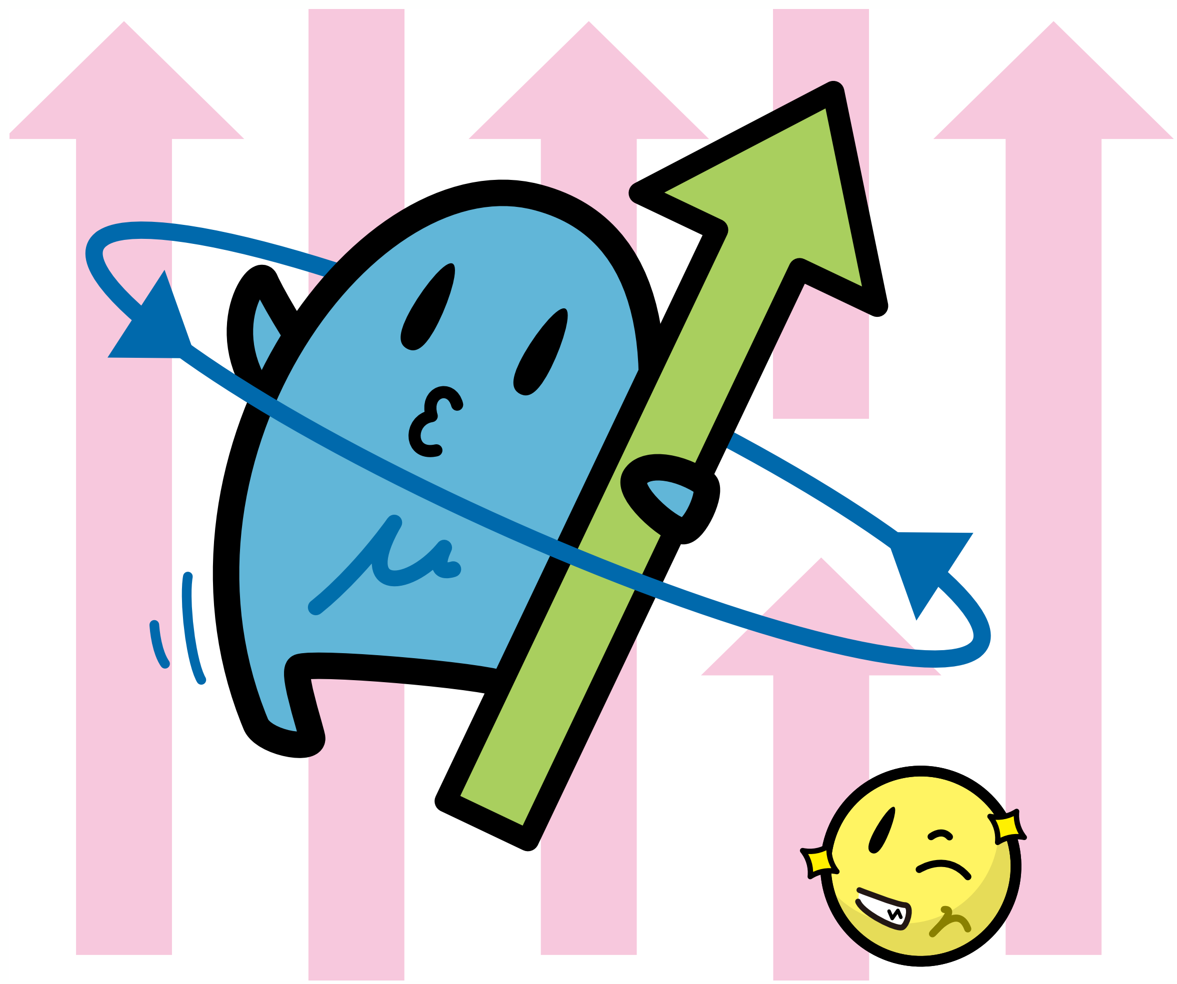


V. Gülpers @ Lattice HVP workshop

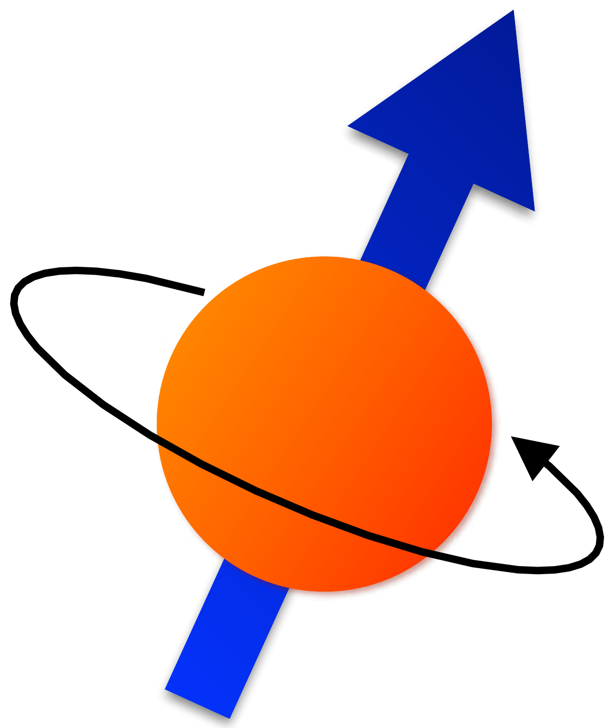


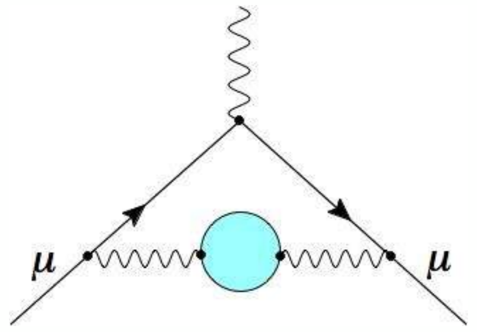
- Some tensions between lattice results for individual contributions.
- Large cancellations between individual contributions: $\delta a_\mu^{\rm IB} \lesssim 1\,\%$
- new, preliminary results by Mainz [Erb, Parrino @ Lattice 2024] agree with RBC/UKQCD.

Outline

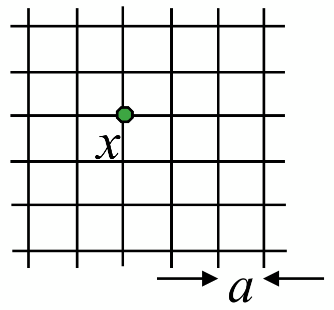


- Introduction
- Introduction to lattice QCD
  - sources of systematic errors
  - state of the art
- Lattice HVP
  - Introduction
  - systematic errors
  - separation prescription
  - Finite Volume (FV) effects
  - long-distance tail
  - windows in Euclidean time
  - Results and updates from Lattice 2024
- Summary and Outlook





Lattice HVP: long-distance tail (again)

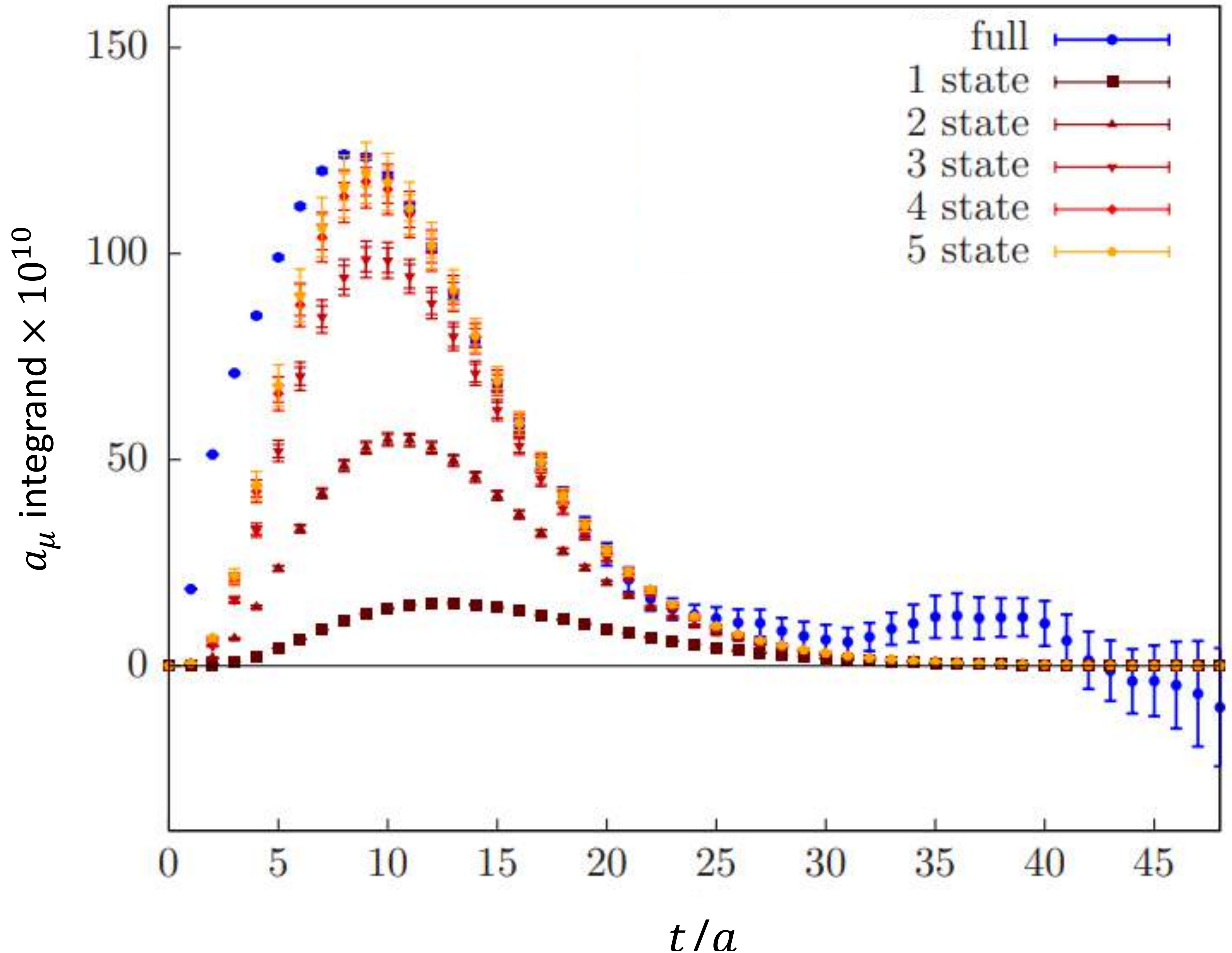


$$C(t) = \frac{1}{3} \sum_{i,x} \langle j_i^{\mathrm{EM}}(x,t)\, j_i^{\mathrm{EM}}(0,0) \rangle$$

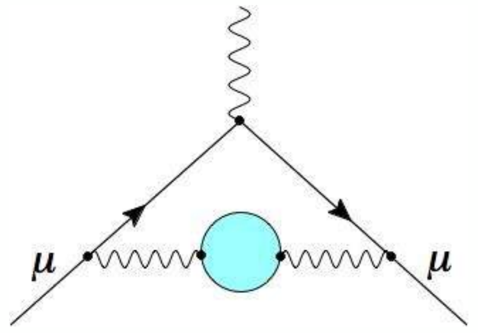
- Start with spectral decomposition: $C(t) = \sum_{n=0}^{\infty} |A_n|^2 e^{-E_n t}$
  - ✦ obtain low-lying finite-volume spectrum (E_n, A_n) in dedicated study using additional operators that couple to two-pion states
  - ✦ use to reconstruct $C(t > t_c)$
  - ✦ can be used to improve bounding method:

    $$C(t) \to C(t) - \sum_{n=0}^{N} A_n^2 e^{-E_n t}$$

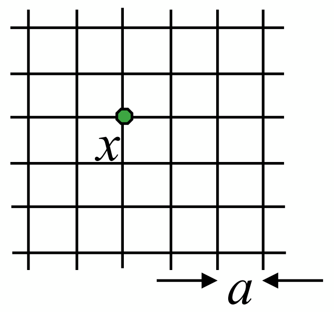
    use E_{N+1} in upper bound
  - ✦ yields big reduction in stat. errors (compared with bounding method)



J. Mckeon @ Lattice 2024 (RBC/UKQCD)
See also:
[Bruno et al, RBC/UKQCD, arXiv:1910.11745]



Lattice HVP: long-distance tail



$$C(t) = \frac{1}{3}\sum_{i,x}\langle j_i^{\mathrm{EM}}(x,t)\, j_i^{\mathrm{EM}}(0,0)\rangle$$

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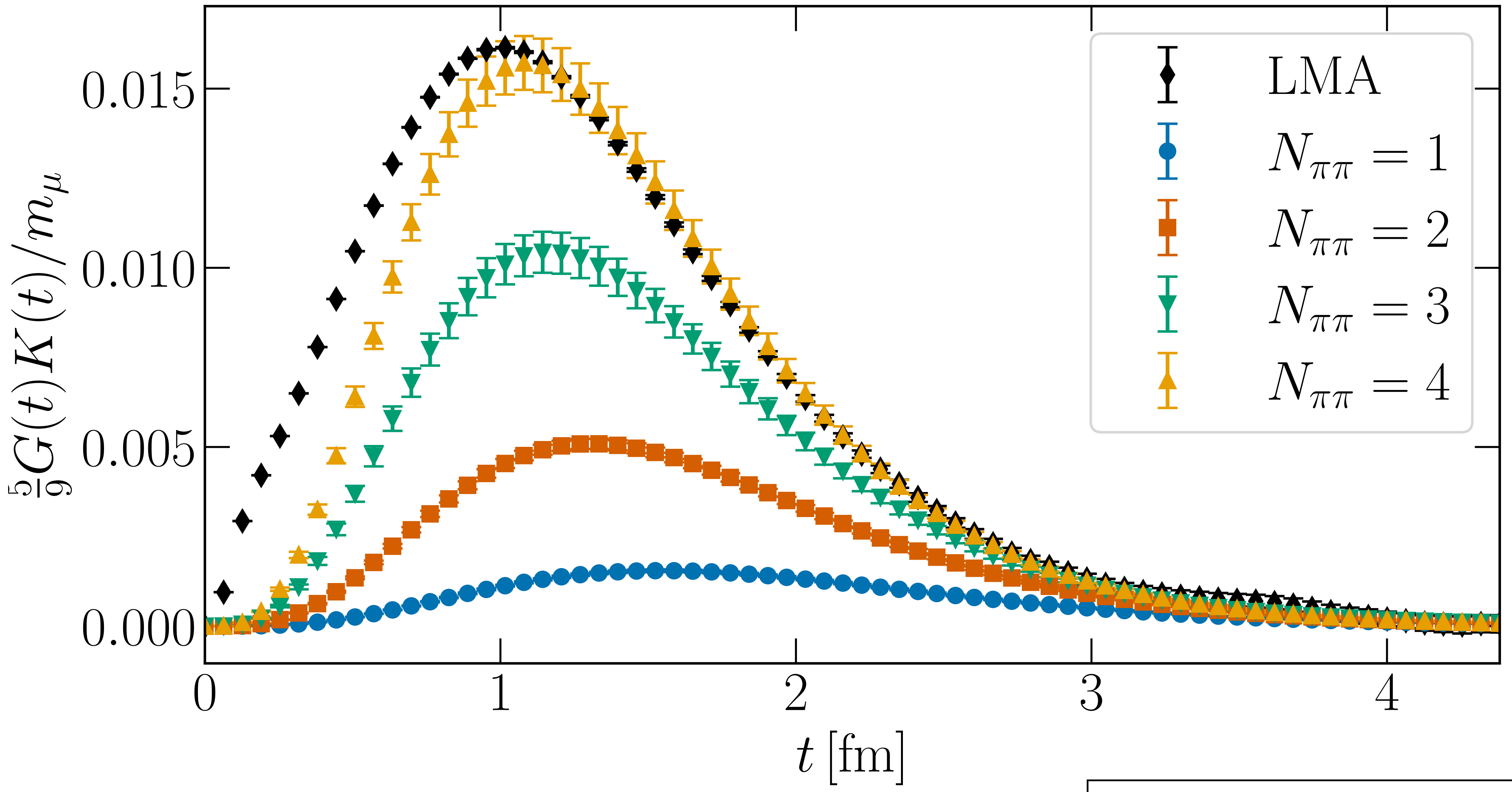
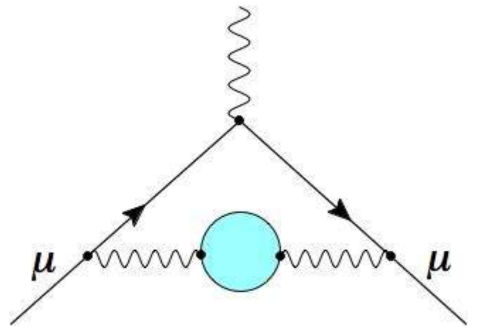
  - ✦ can be used to improve bounding method:

    $$C(t) \to C(t) - \sum_{n=0}^{N} A_n^2\, e^{-E_n t}$$

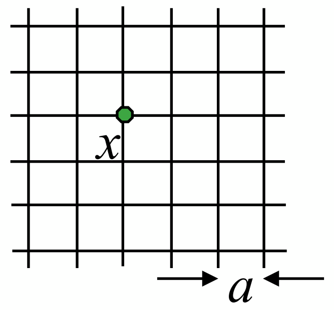
    use E_{N+1} in upper bound
  - ✦ yields $\sim \times 2$ reduction in stat. errors



S. Kuberski & N. Miller @ Lattice 2024 (Mainz)
See also: A. Gerardin et al, PRD 2019



Lattice HVP: long-distance tail



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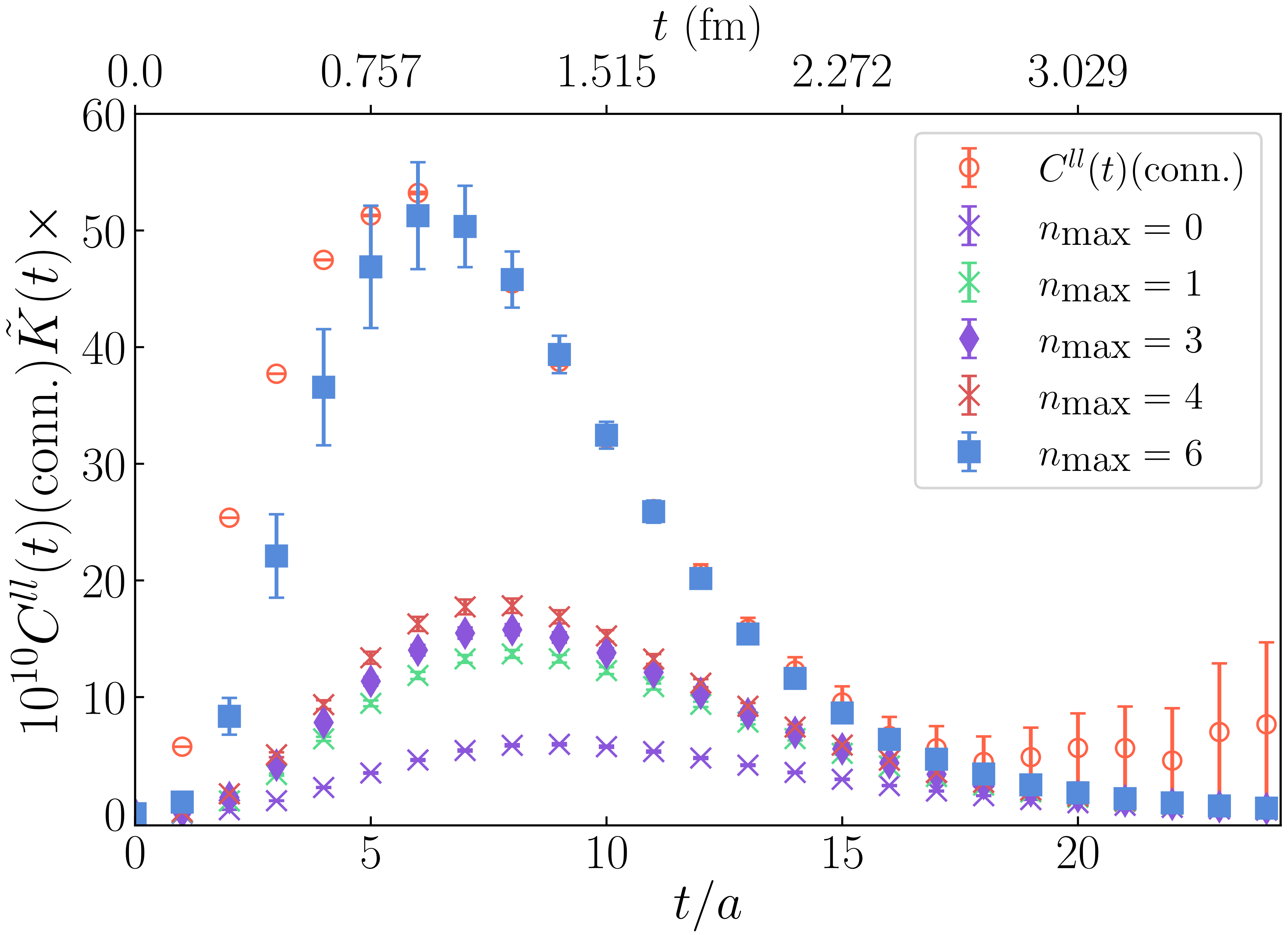
  - ✦ obtain low-lying finite-volume spectrum (E_n, A_n) in dedicated study using additional operators that couple to two-pion states
  - ✦ use to reconstruct $C(t > t_c)$
  - ✦ can be used to improve bounding method:

    $$C(t) \to C(t) - \sum_{n=0}^{N} A_n^2 e^{-E_n t}$$

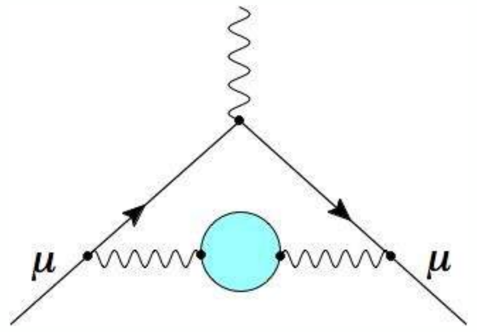
    use E_{N+1} in upper bound
  - ✦ yields $\sim \times 2.5$ reduction in stat. errors (compared with bounding method)



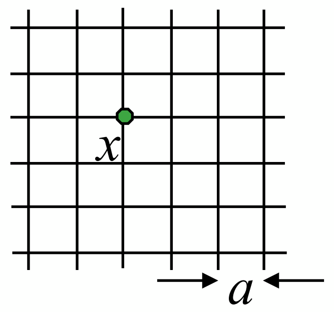
[Lahert, et al, arXiv:2409.00756]
see also:
[Lahert et al, arXiv:2112.11647]

First LQCD calculation with staggered multi-pion operators

see also:
[Frech for BMW @ Lattice 2024]



Windows in Euclidean time



$$a_\mu^{\mathrm{HVP,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, \tilde{w}(t)\, C(t)$$

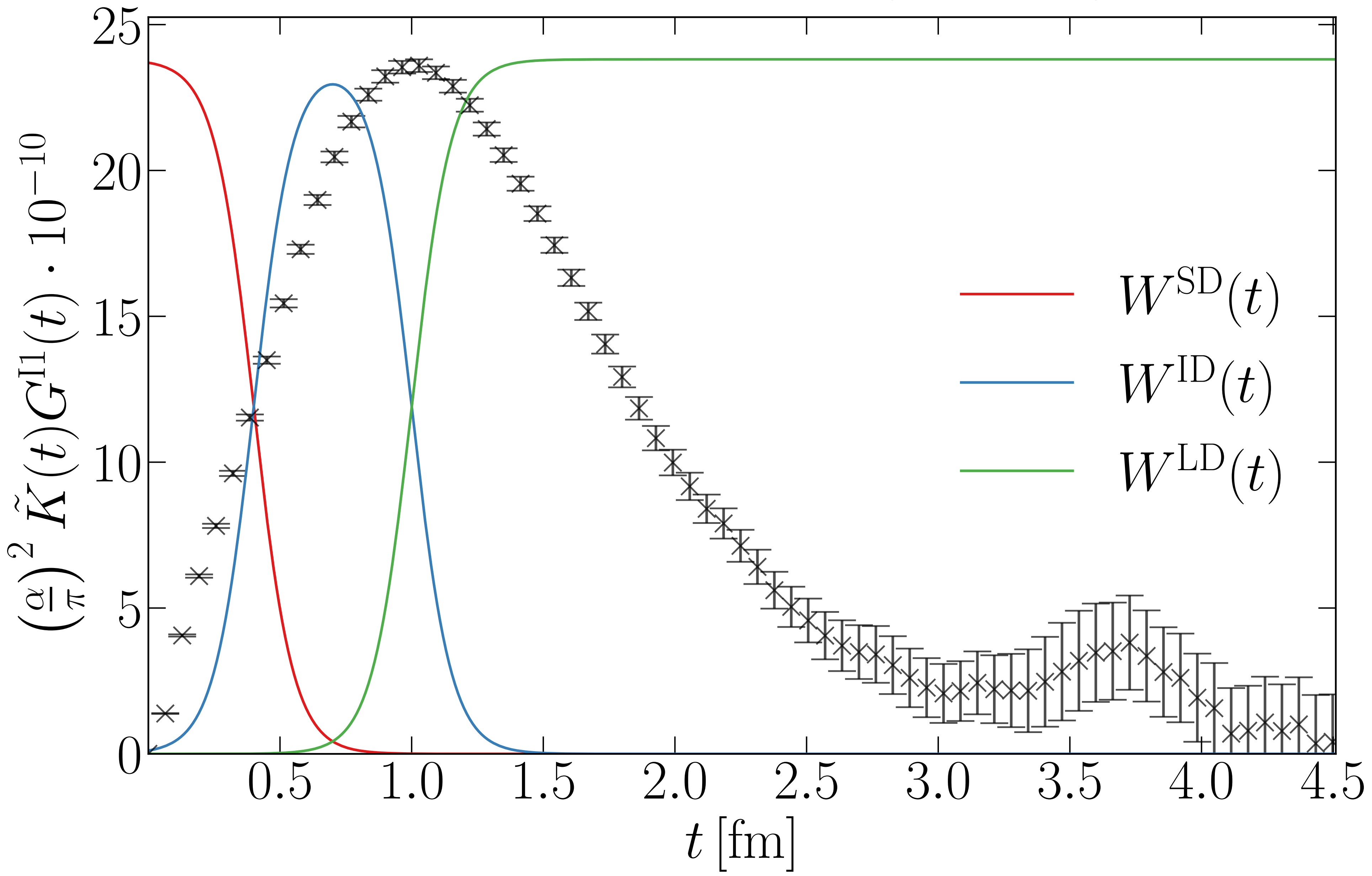
- Use windows in Euclidean time to consider the different time regions separately

  [T. Blum et al, arXiv:1801.07224, 2018 PRL]

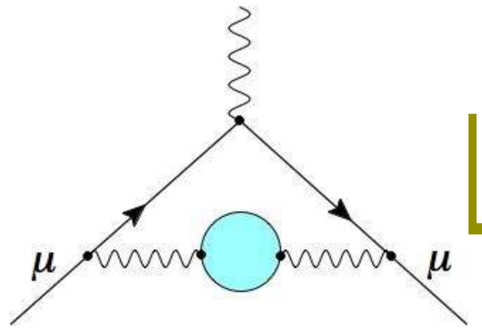
$$t_0 = 0.4\,\mathrm{fm}, t_1 = 1.0\,\mathrm{fm}$$

Short Distance (SD)	$t : 0 \to t_0$
Intermediate (W)	$t : t_0 \to t_1$
Long Distance (LD)	$t : t_1 \to \infty$

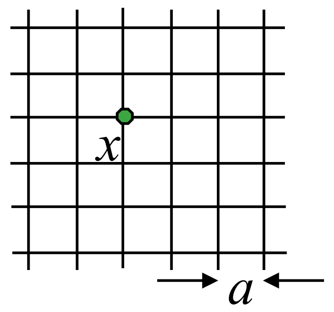
S. Kuberski @ Lattice 2023



- disentangle systematics/statistics from long distance/FV and discretization effects
- intermediate window: easy to compute in lattice QCD; compare to disperse approach
- Internal cross check: compute each window separately (in continuum, infinite volume limits,…) and combine: $a_\mu = a_\mu^{\mathrm{SD}} + a_\mu^{\mathrm{W}} + a_\mu^{\mathrm{LD}}$



Lattice HVP: intermediate window (W)

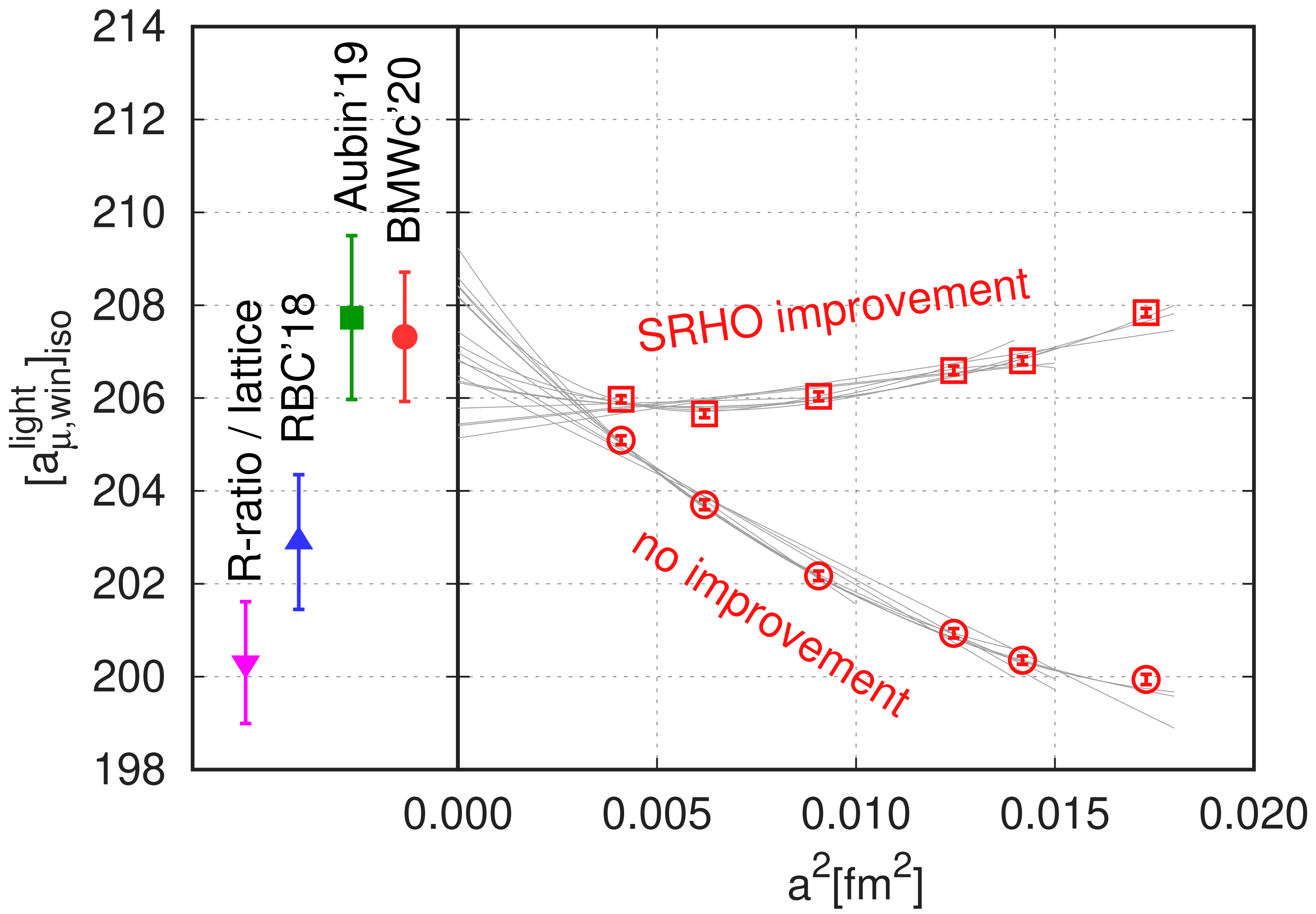


In 2020 WP:

- Lattice HVP average at $2.6\,\%$ total uncertainty: $a_\mu^{\text{HVP,LO}} = 711.6\,(18.4) \times 10^{10}$
- BMW 20 [Sz. Borsanyi et al, arXiv:2002.12347, 2021 Nature] first LQCD calculation with sub-percent $(0.8\,\%)$ error in tension with data-driven HVP (2.1σ)
- Further tensions for intermediate window:

  -3.7σ tension with data-driven evaluation

  -2.2σ tension with RBC/UKQCD18



Staggered fermions:

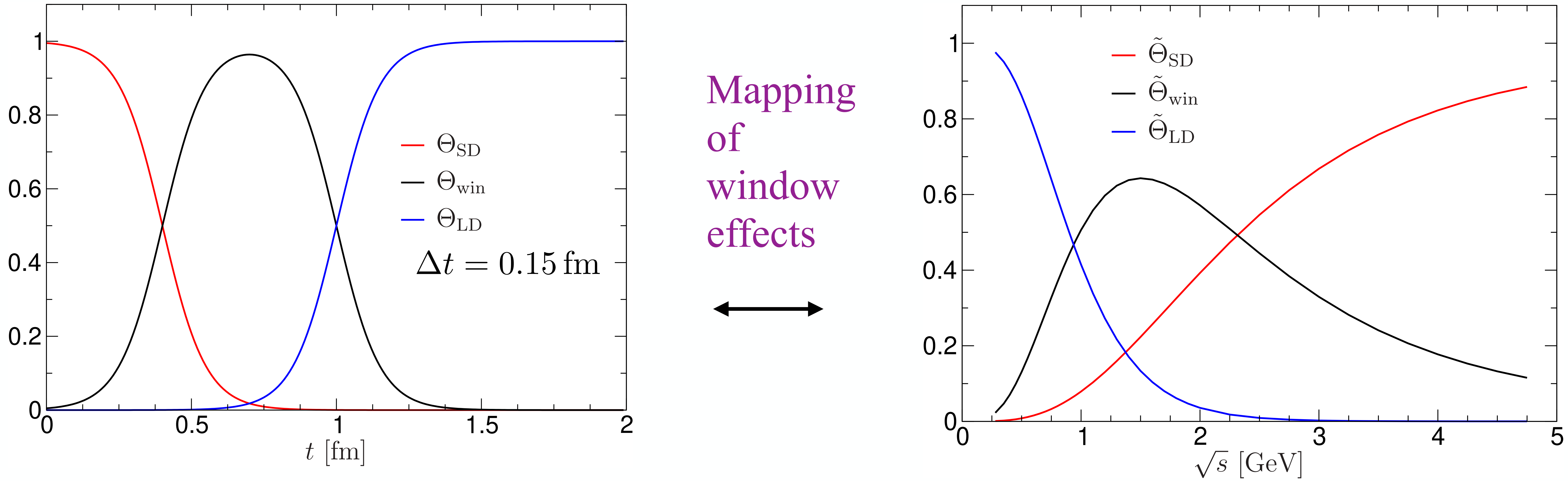
- taste-breaking effects (which yield taste splittings) are significant (sometimes dominant) source of discretization errors
- possible to use EFT schemes (ChPT, Chiral Model, MLLGS) to correct for taste-splitting effects before taking continuum extrapolation: continuum limit should not be affected

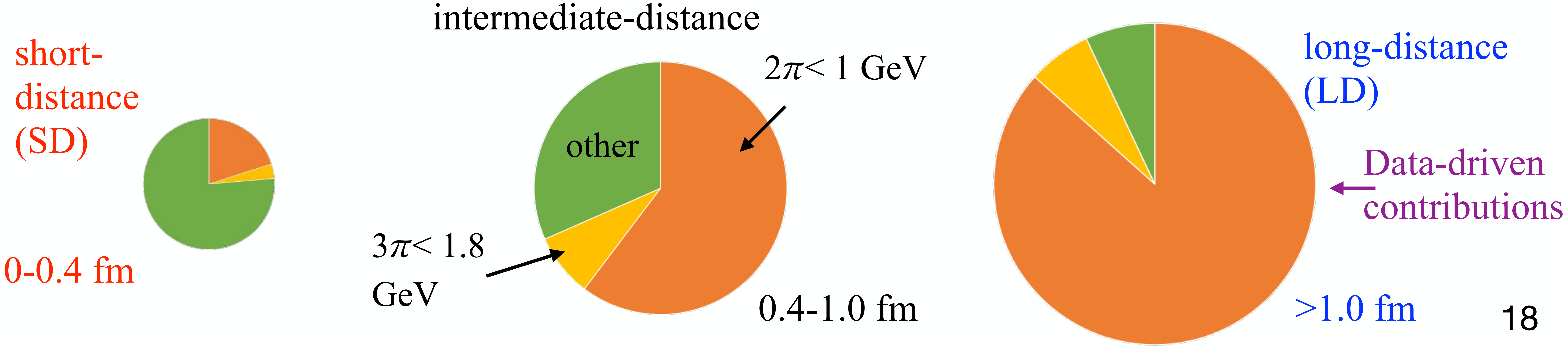
cross section inputs to windows observables

C. Davies @ Lattice 2024

Comparing data-driven and lattice HVP results

Colangelo et al
2205.12963





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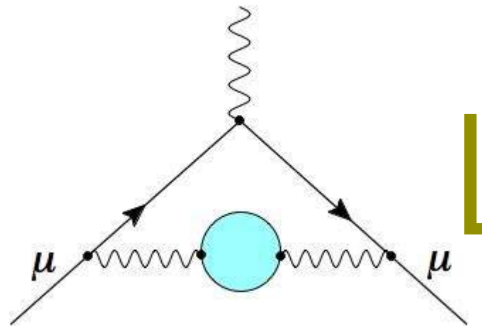
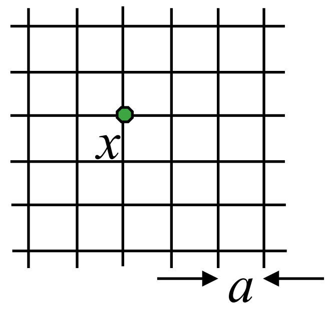
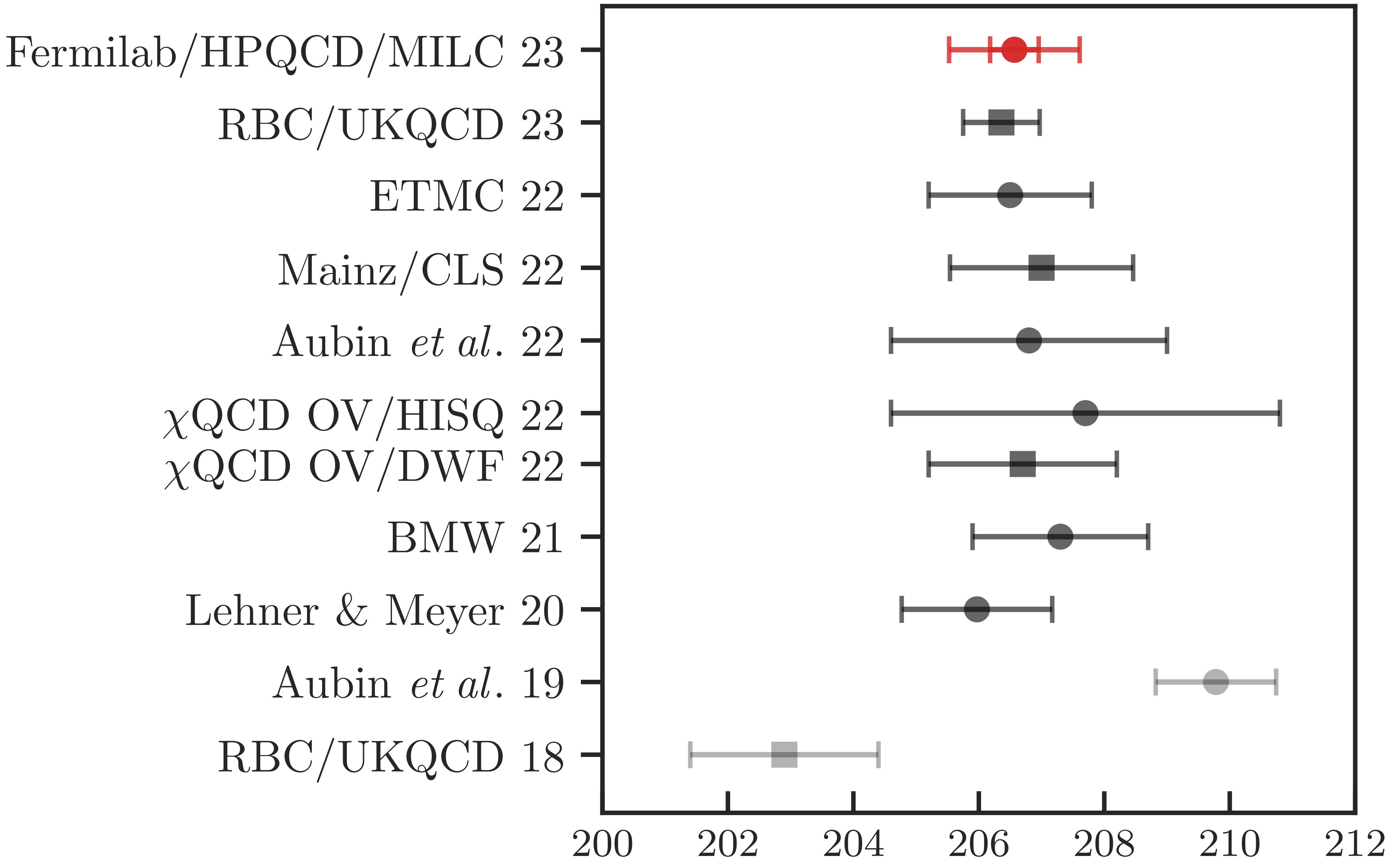


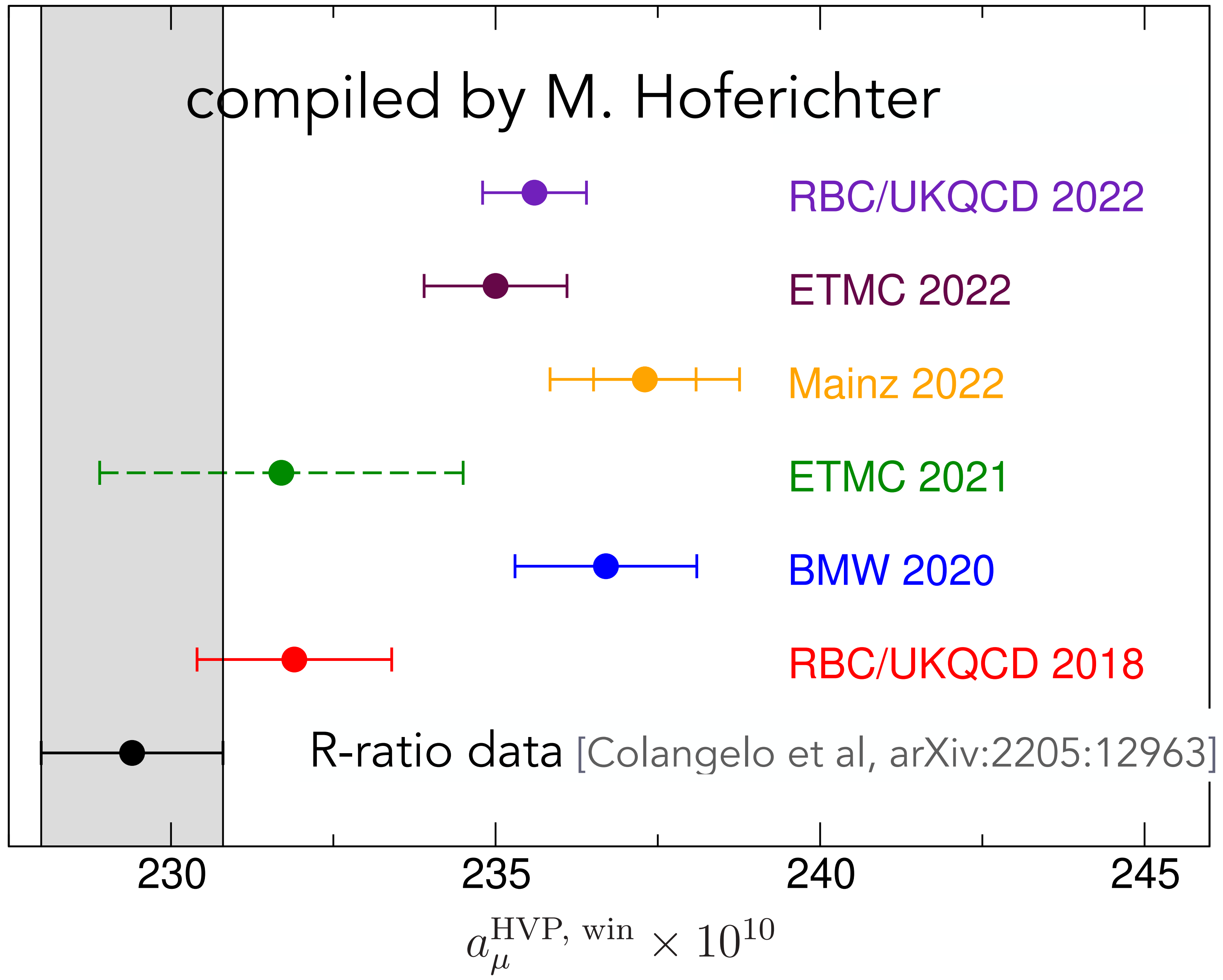
Lattice HVP: intermediate window (W)



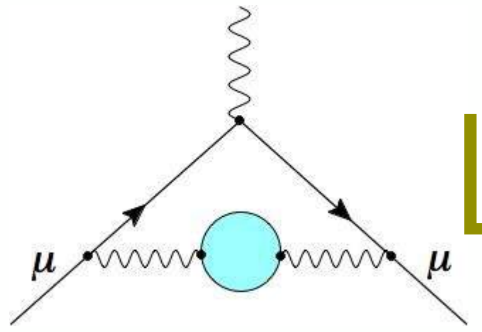
- new results in 2022/2023 for intermediate window, a_μ^W from six different lattice groups.
- blind analyses: Fermilab/HPQCD/MILC + RBC/UKQCD
- lattice-only comparison of light-quark connected contribution to intermediate window:



- LQCD vs dispersive evaluation of intermediate window observable



$\sim (3.5 - 4)\sigma$ tensions between LQCD and (pre-2023) data-driven evaluations



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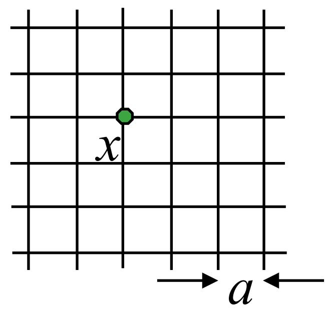

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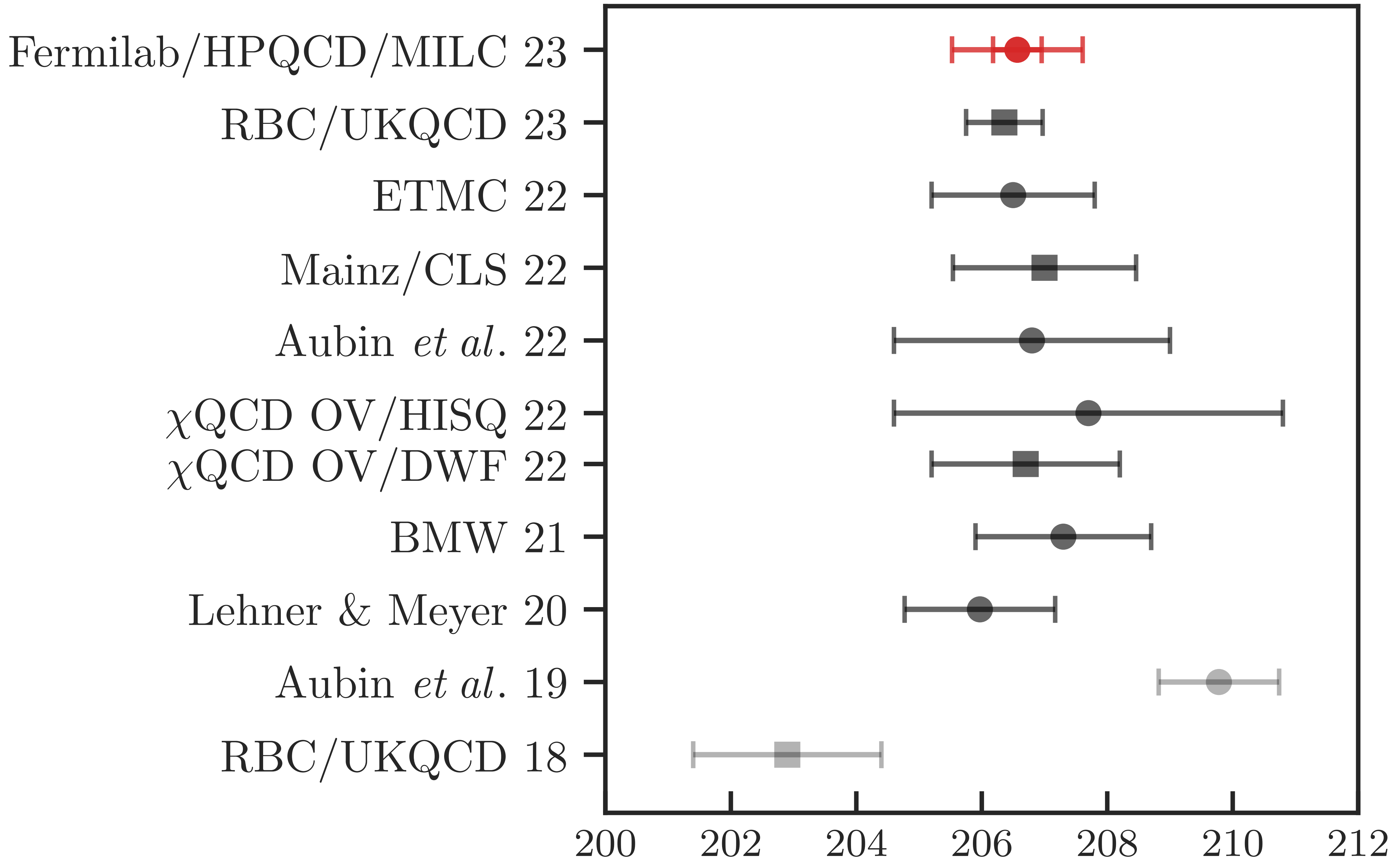
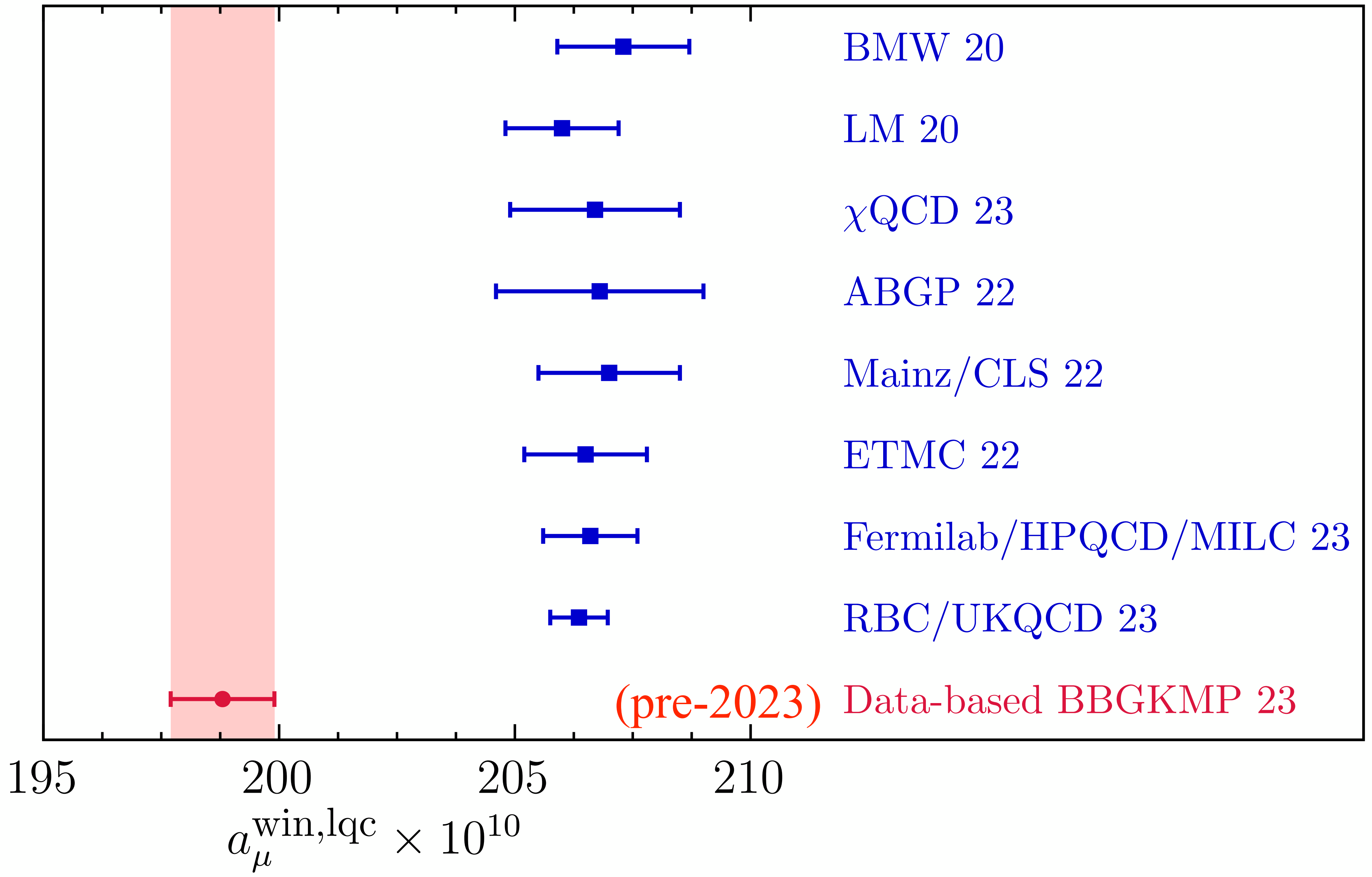
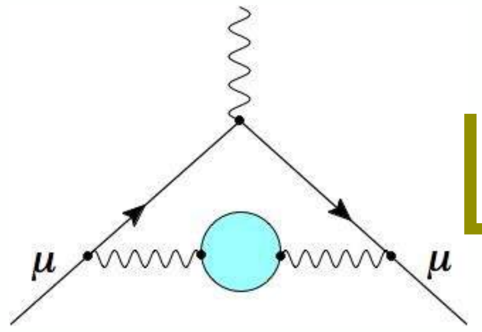


- dispersive evaluation of light-quark connected contribution [G. Benton, et al, arXiv:2306.16808]*

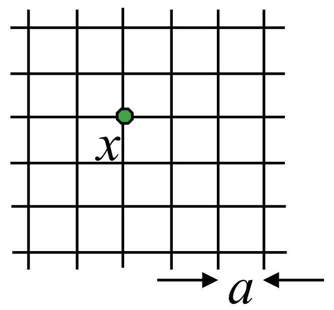


$\sim (3.5 - 4)\sigma$ tensions between LQCD and (pre-2023) data-driven evaluations

*employs disp. results for IB [Hoferichter et al, arXiv:2208.08993]



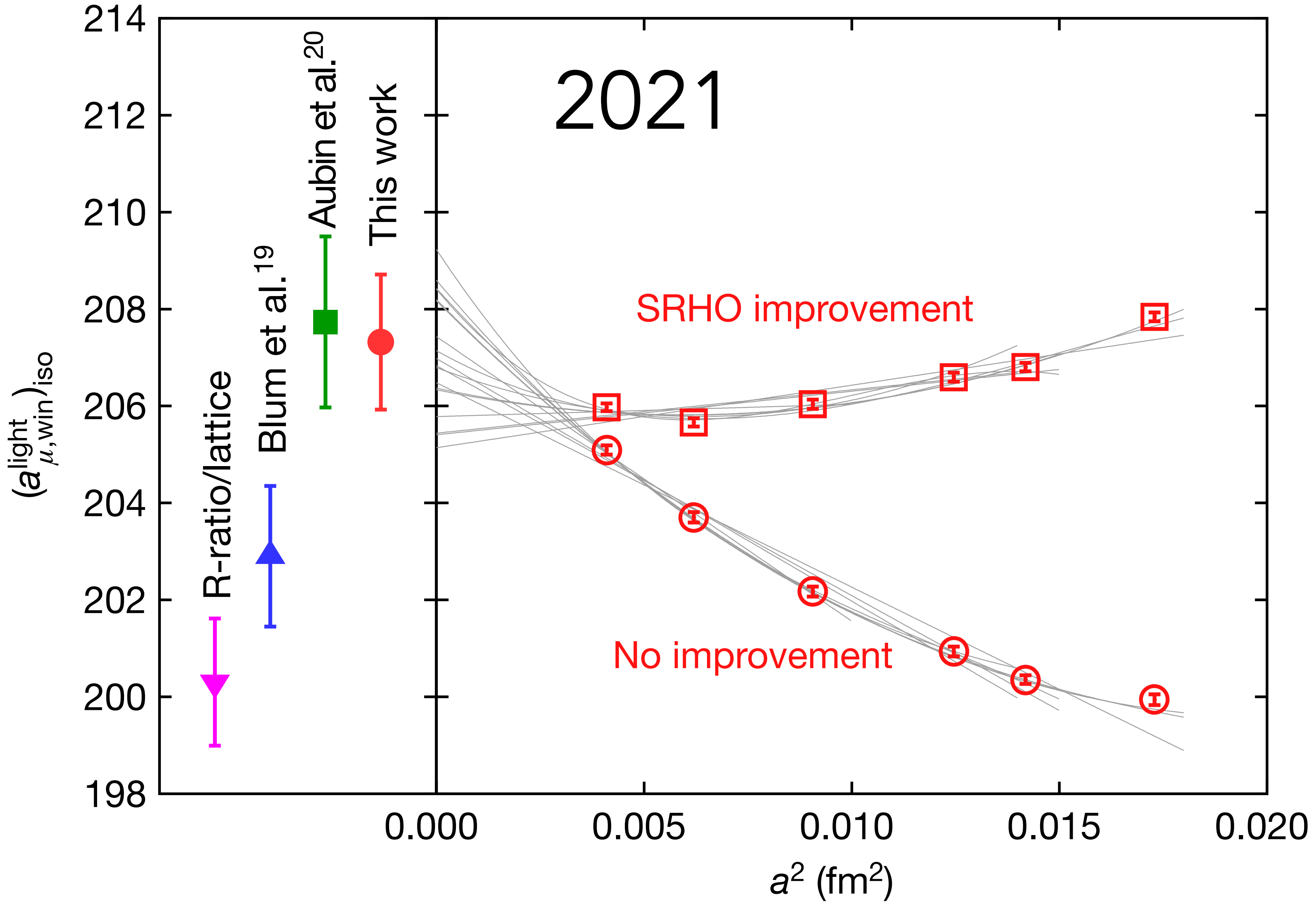
Lattice HVP: intermediate window (W)



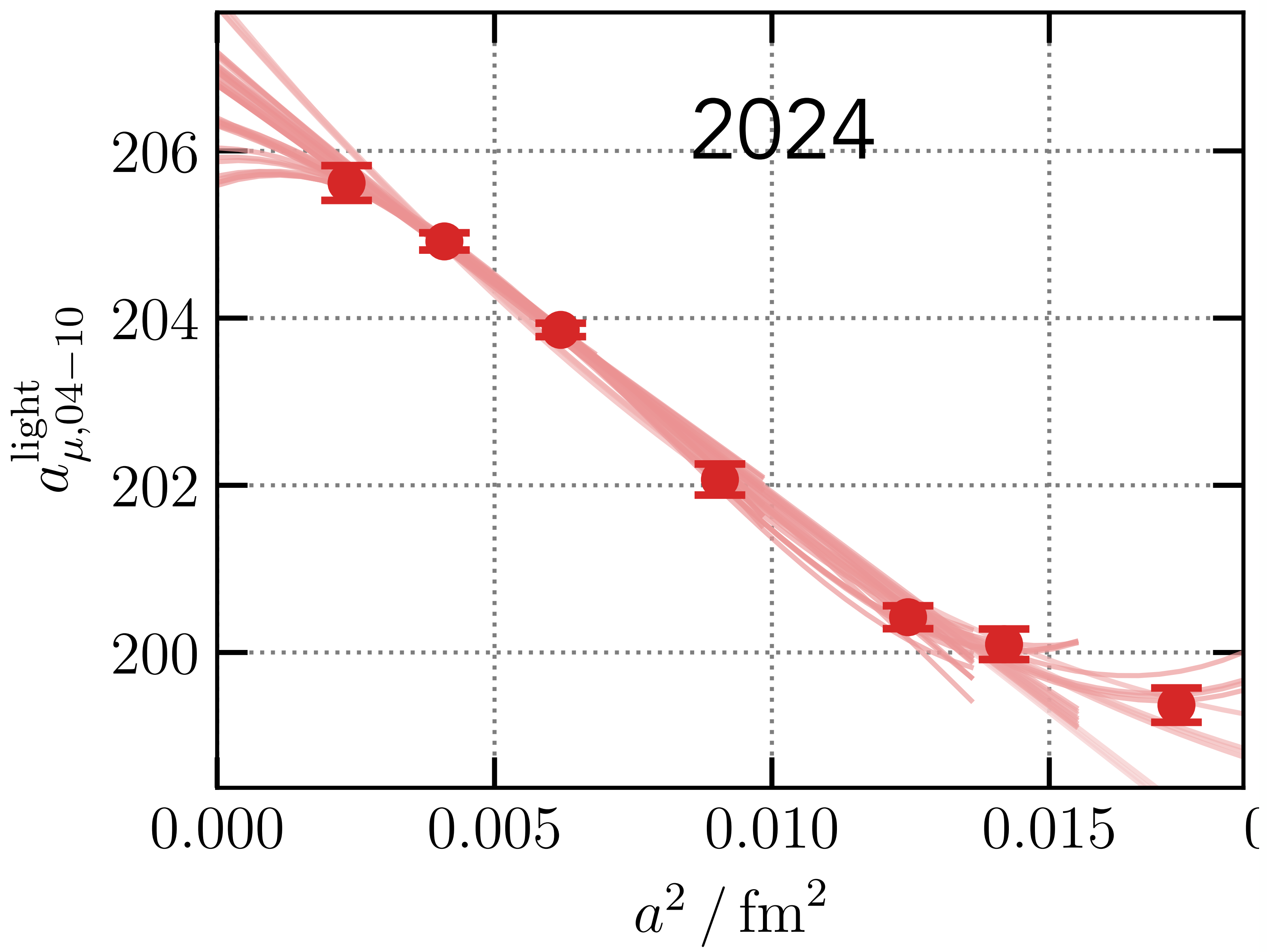
update: BMW+DMZ 2024

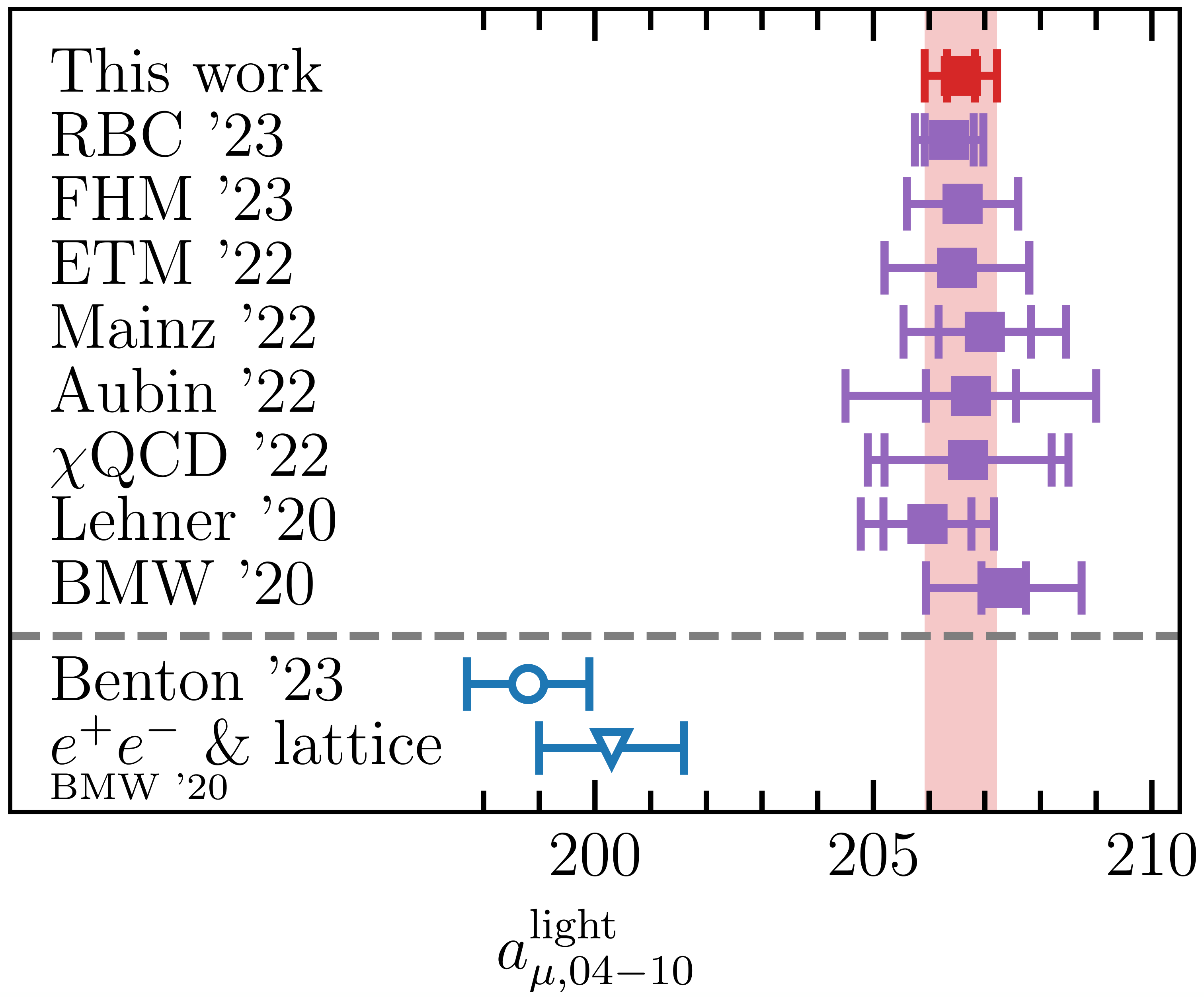
- intermediate window, a_μ^W

- BMW 20 [Sz. Borsanyi et al, arXiv:2002.12347, 2021 Nature]



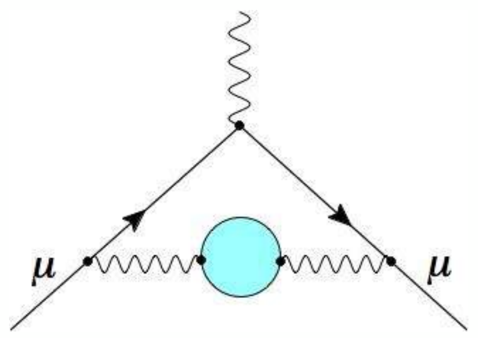
- BMW-DMZ 24 [A. Boccaletti et al, arXiv:2407.10913]



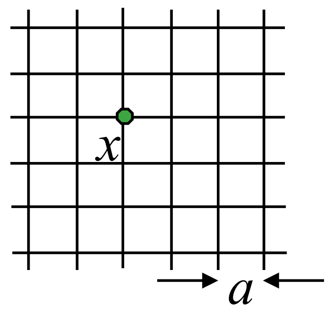


- Improvement:
  correct lattice data for discretization effects due to taste-splittings before taking continuum limit.

Continuum extrapolations obtained only from data not corrected for taste-splittings.



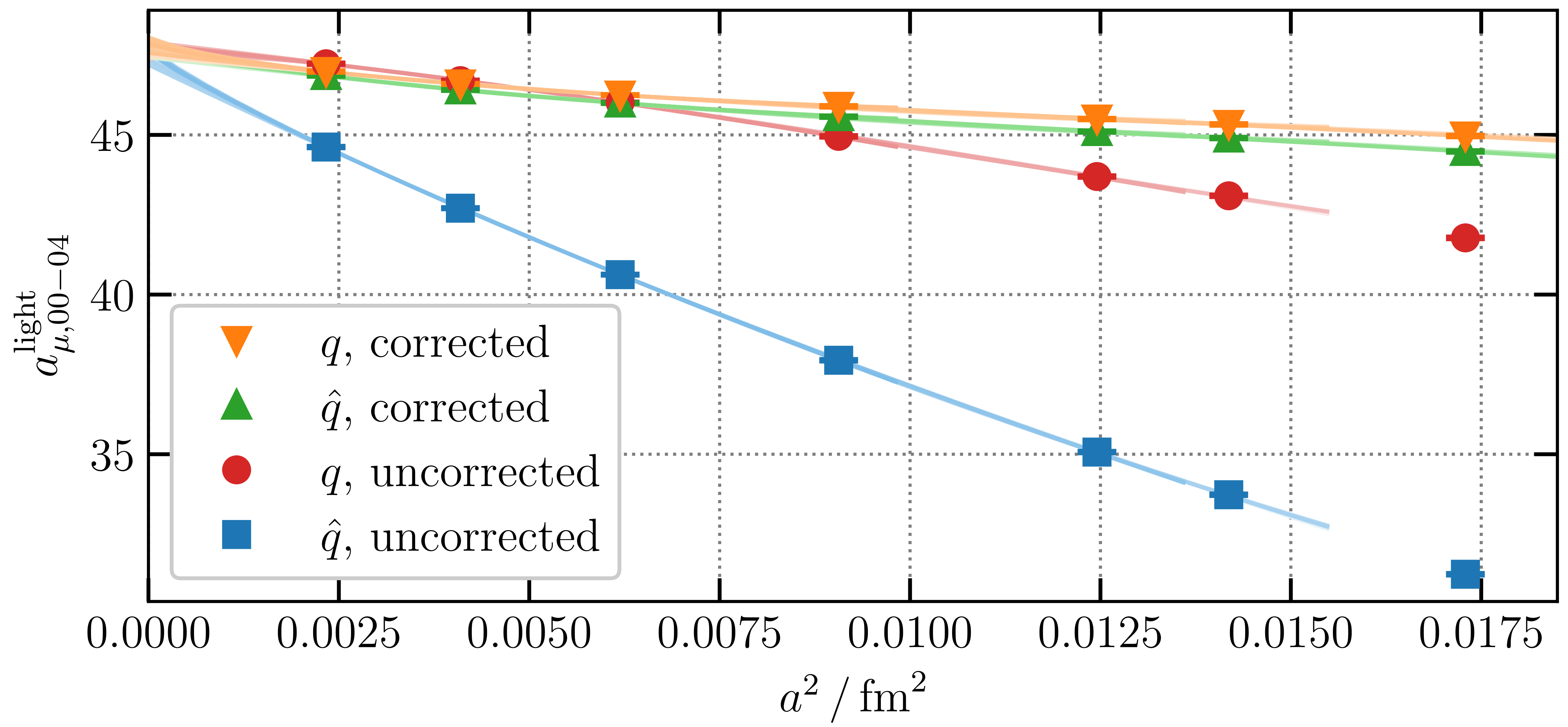
Lattice HVP: short-distance window (SD)



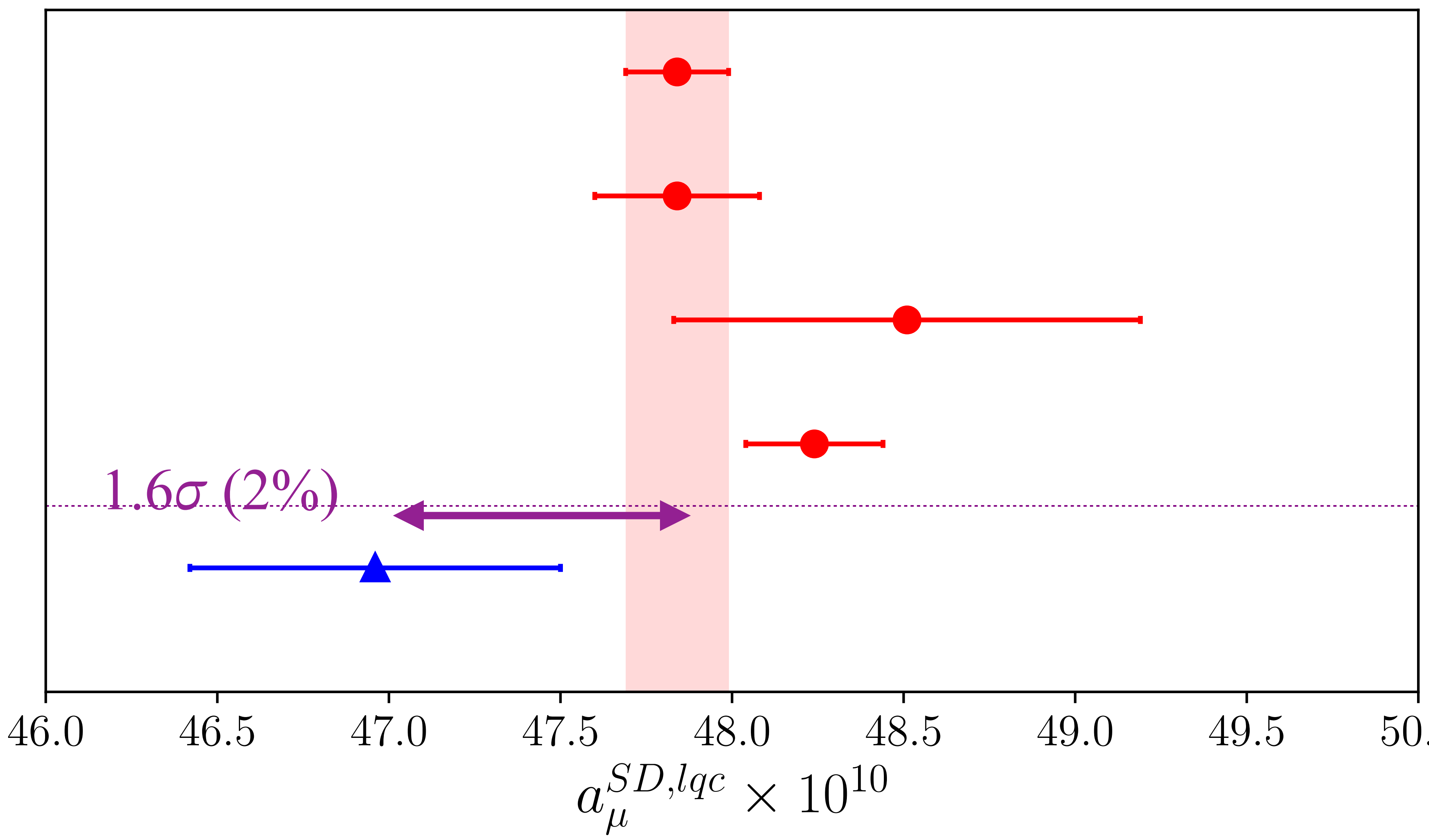
update: BMW+DMZ 2024

- short-distance window, a_μ^{SD}

- BMW 24 [A. Boccaletti et al, arXiv:2407.10913]



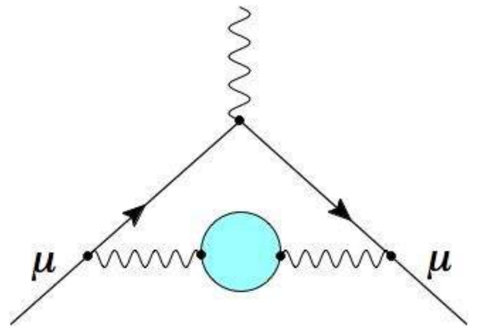
C. Davies @ Lattice 2024



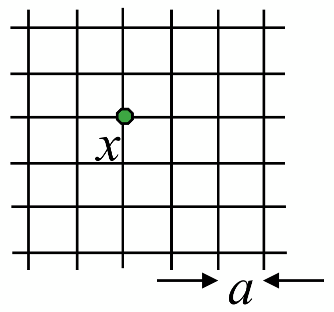
- corrected: remove log-enhanced discretization effects at tree-level

- $\hat{q}$: $\left[t^2 - \frac{4}{(aQ)^2}\sin^2\left(\frac{aQt}{2}\right)\right] \rightarrow \left[t^2 - \frac{4}{(a\hat{Q})^2}\sin^2\left(\frac{aQt}{2}\right)\right]$

small tension in SD with pre-2023 data-driven evaluation

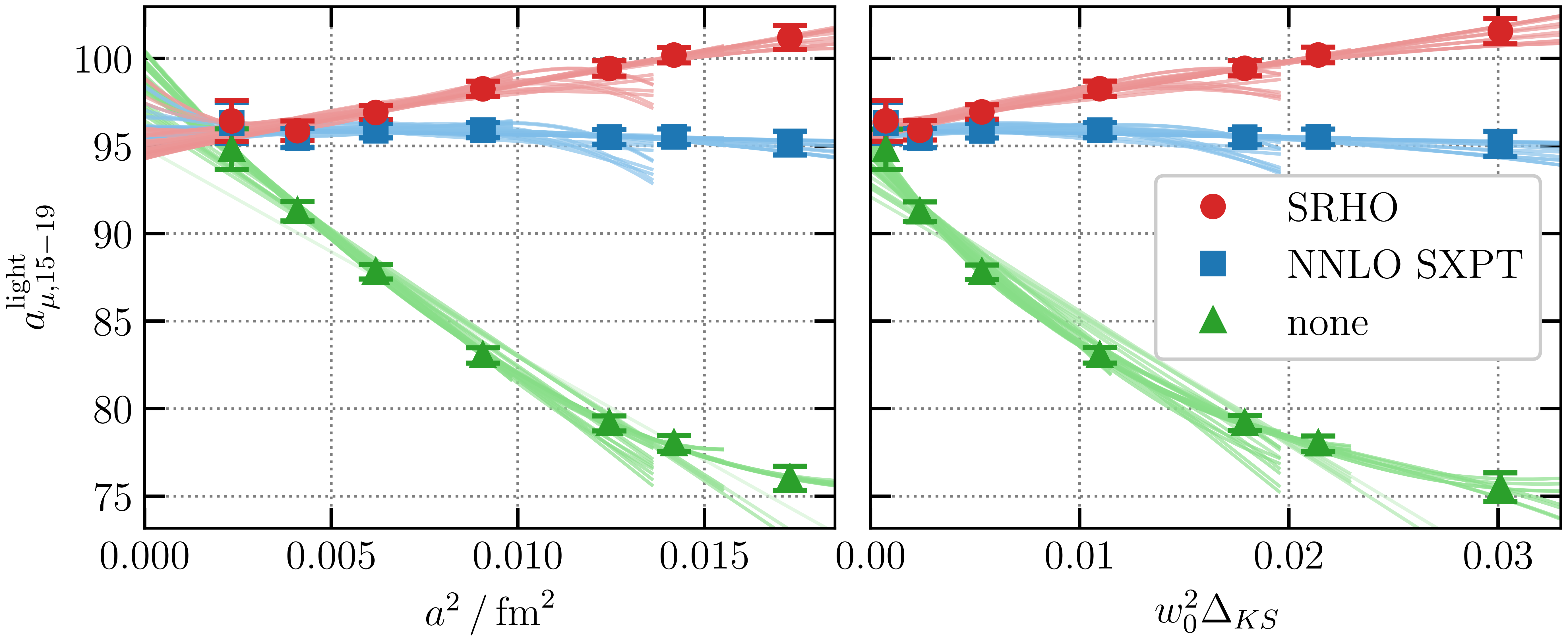


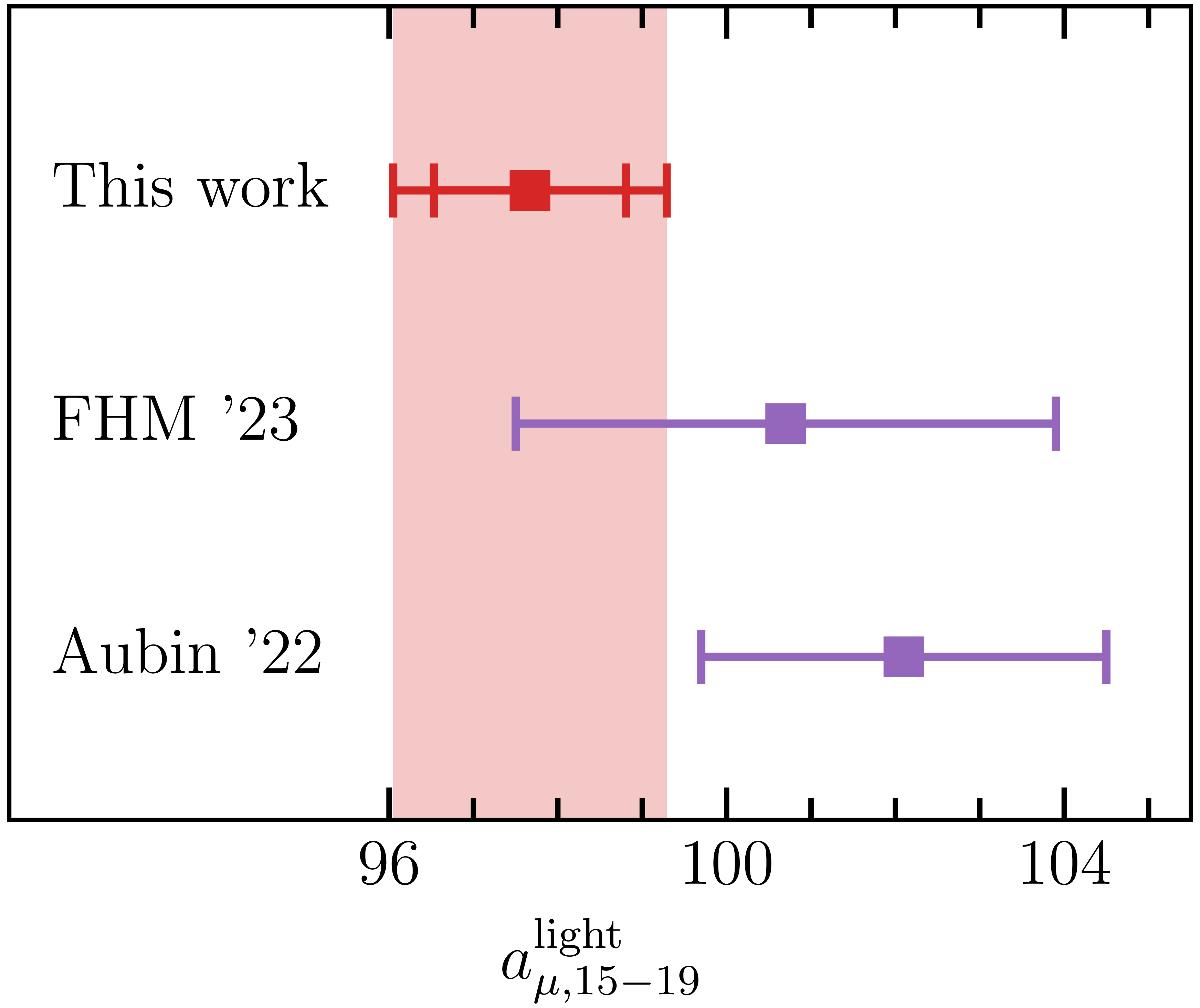
Lattice HVP: 2nd window (W2)



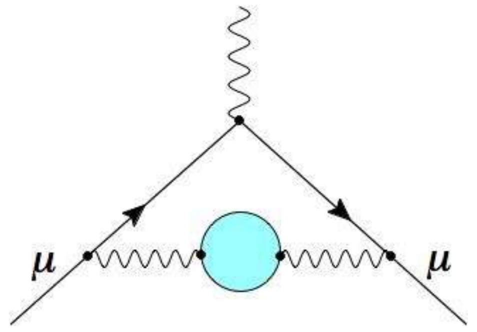
update: BMW+DMZ 2024

- [A. Boccaletti et al, arXiv:2407.10913]

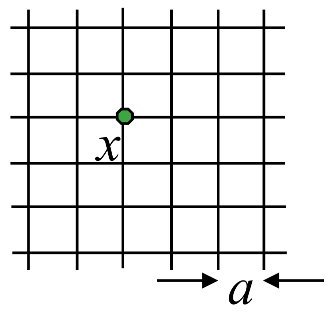




Continuum extrapolations of data with ``No improvement" (green) excluded from model average.



Lattice HVP: windows



update: Fermilab/HPQCD/MILC 2024

FNAL/HPQCD/MILC @ Lattice 2024:

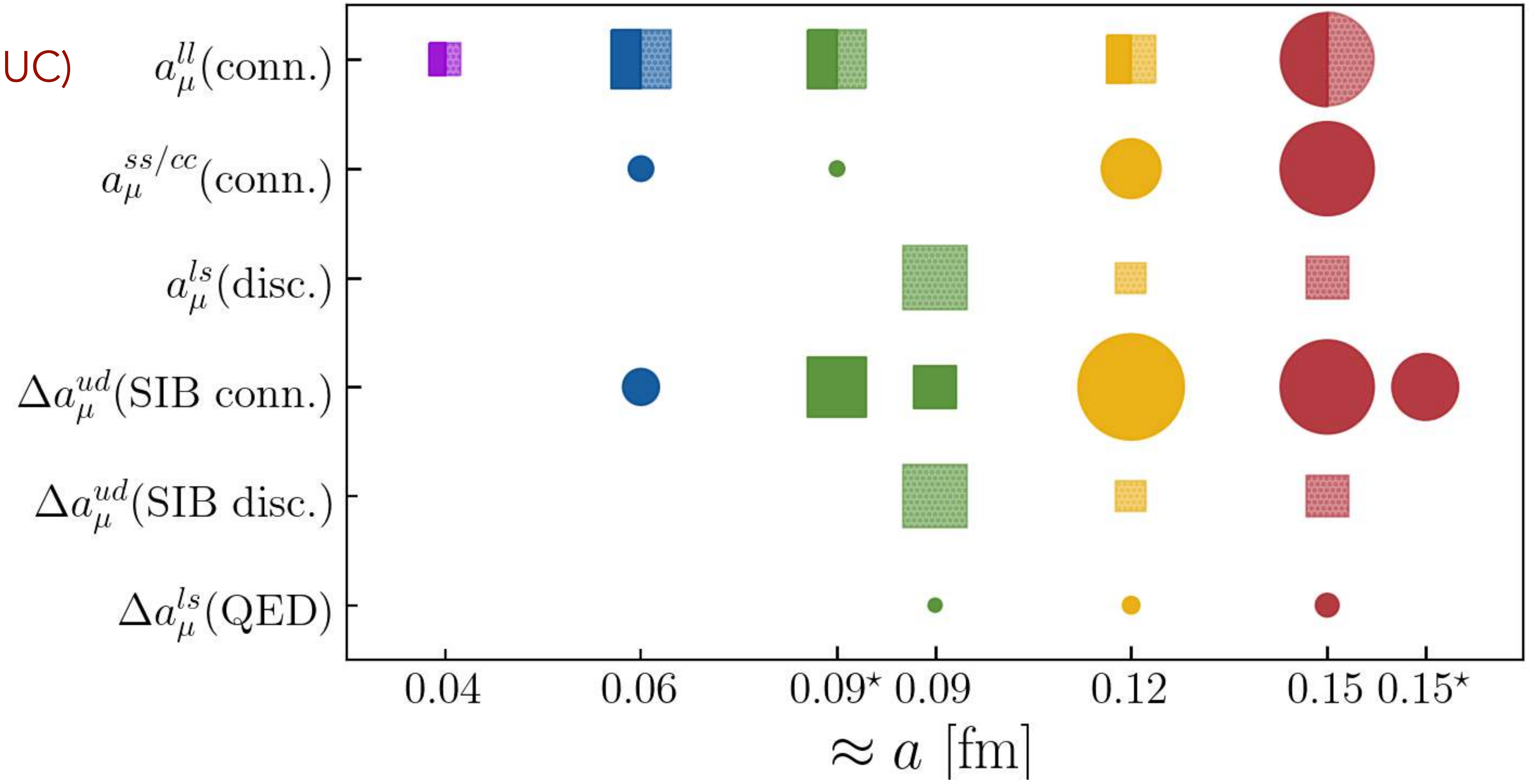
Shaun Lahert (Utah) & Michael Lynch (UIUC)

Shaun Lahert (Utah)

David Clarke (Utah)

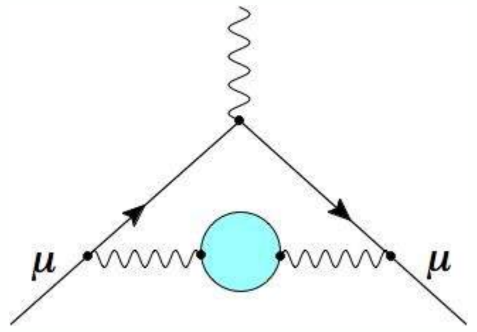
Jake Sitison (U Colorado)

Craig McNeile (Plymouth)

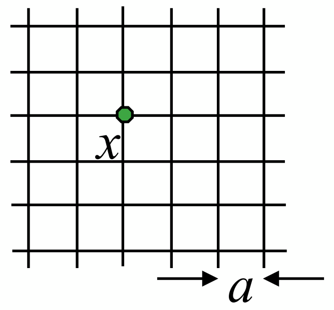


All results are still blinded

- Solid color (local current) hatched (one-link)
- Squares: low-mode improved.
- Size ~ statistics



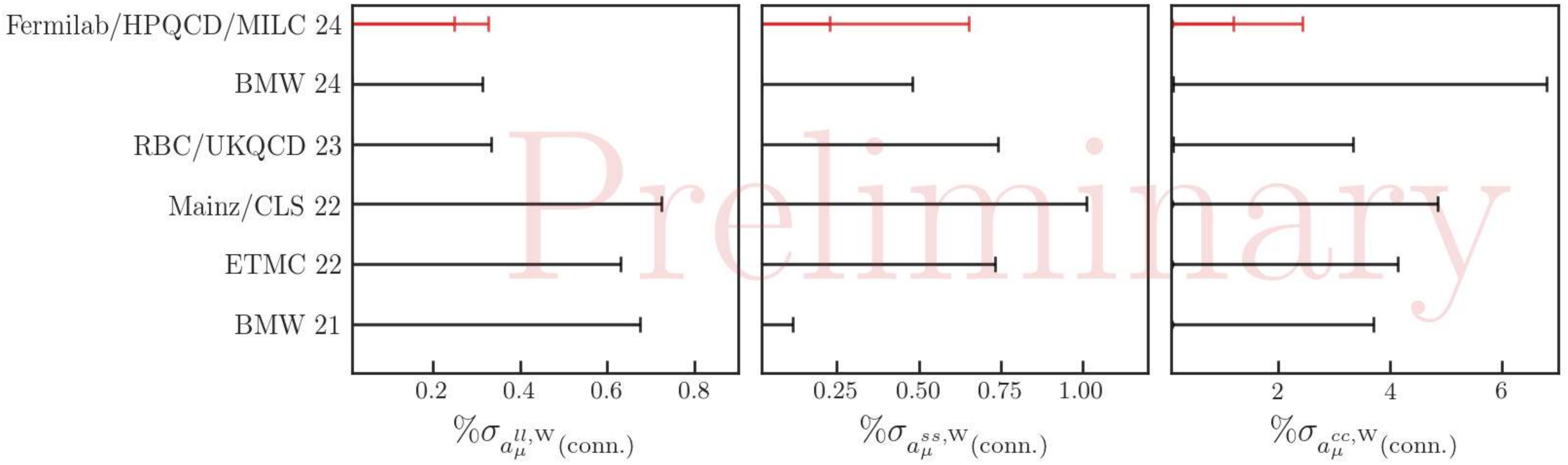
Lattice HVP: windows



update: Fermilab/HPQCD/MILC 2024

S. Lahert for FNAL/HPQCD/MILC @ Lattice 2024

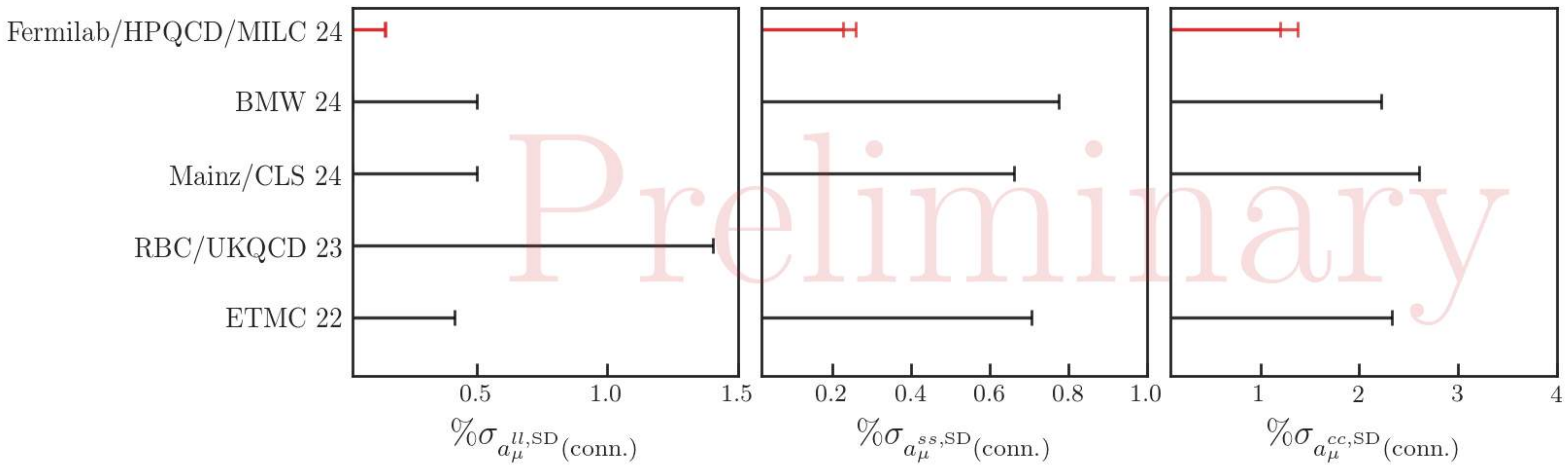
- intermediate window, a_μ^W



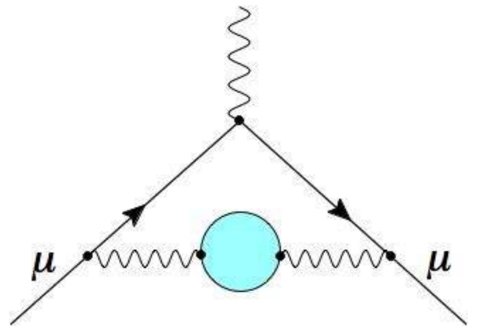
- Scale setting (w_0 fm) is significant uncertainty in all contributions (Inner error bar: no abs. scale setting uncertainty.).

➠ S. Lahert talk @ KEK workshop:
Unblinded results for (all) contributions to a_μ^{SD} and a_μ^W (including correlations).

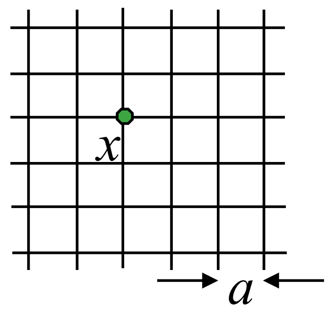
- short-distance window, a_μ^{SD}



- Competitive uncertainties for all flavors.
- HISQ local-current mitigating log-enhancement.



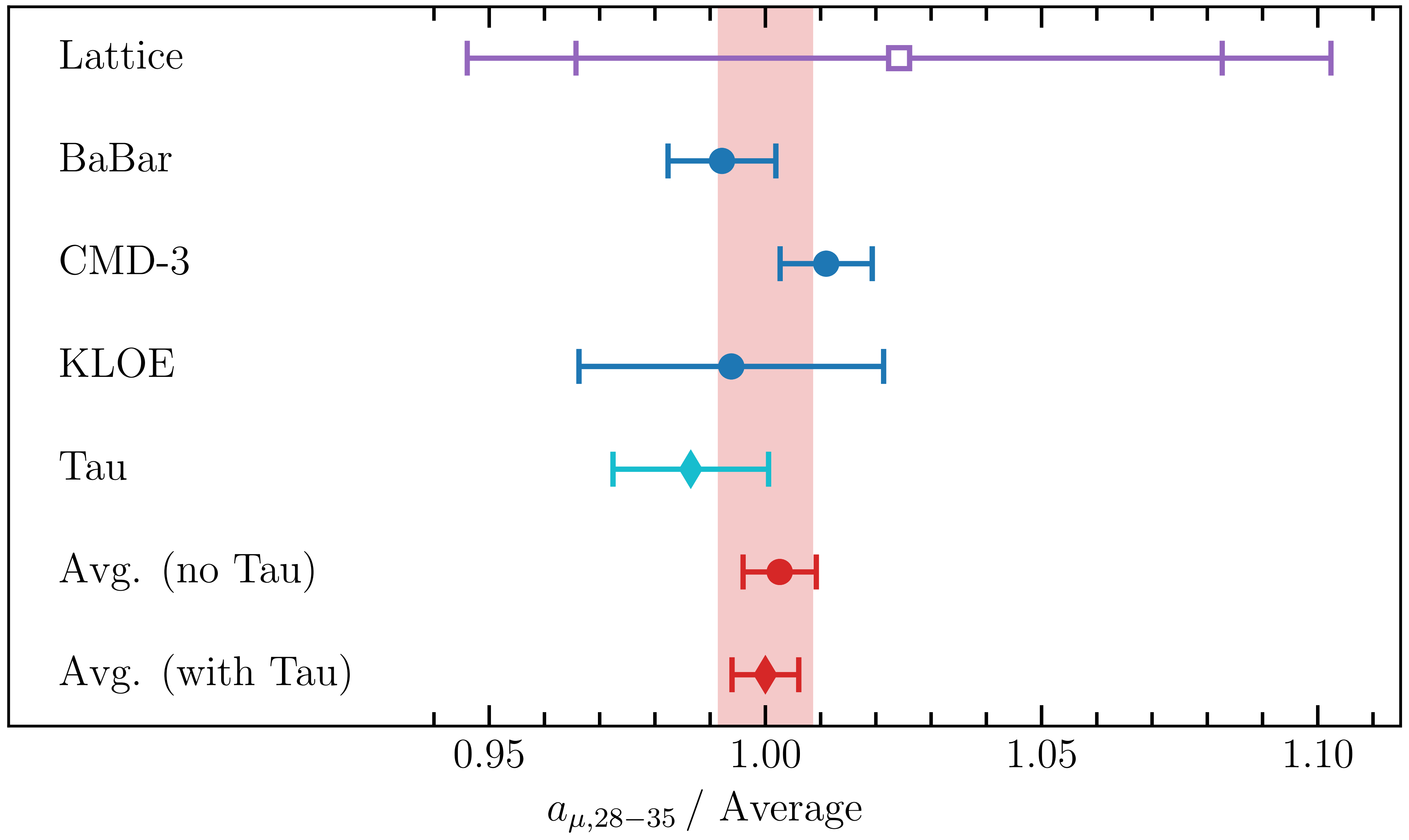
Lattice HVP: long-distance window



update: BMW+DMZ 2024

- BMW+DMZ 24 [A. Boccaletti et al, arXiv:2407.10913]

- statistical/systematic errors at long distances, $t \gtrsim 2.5$ fm still large:

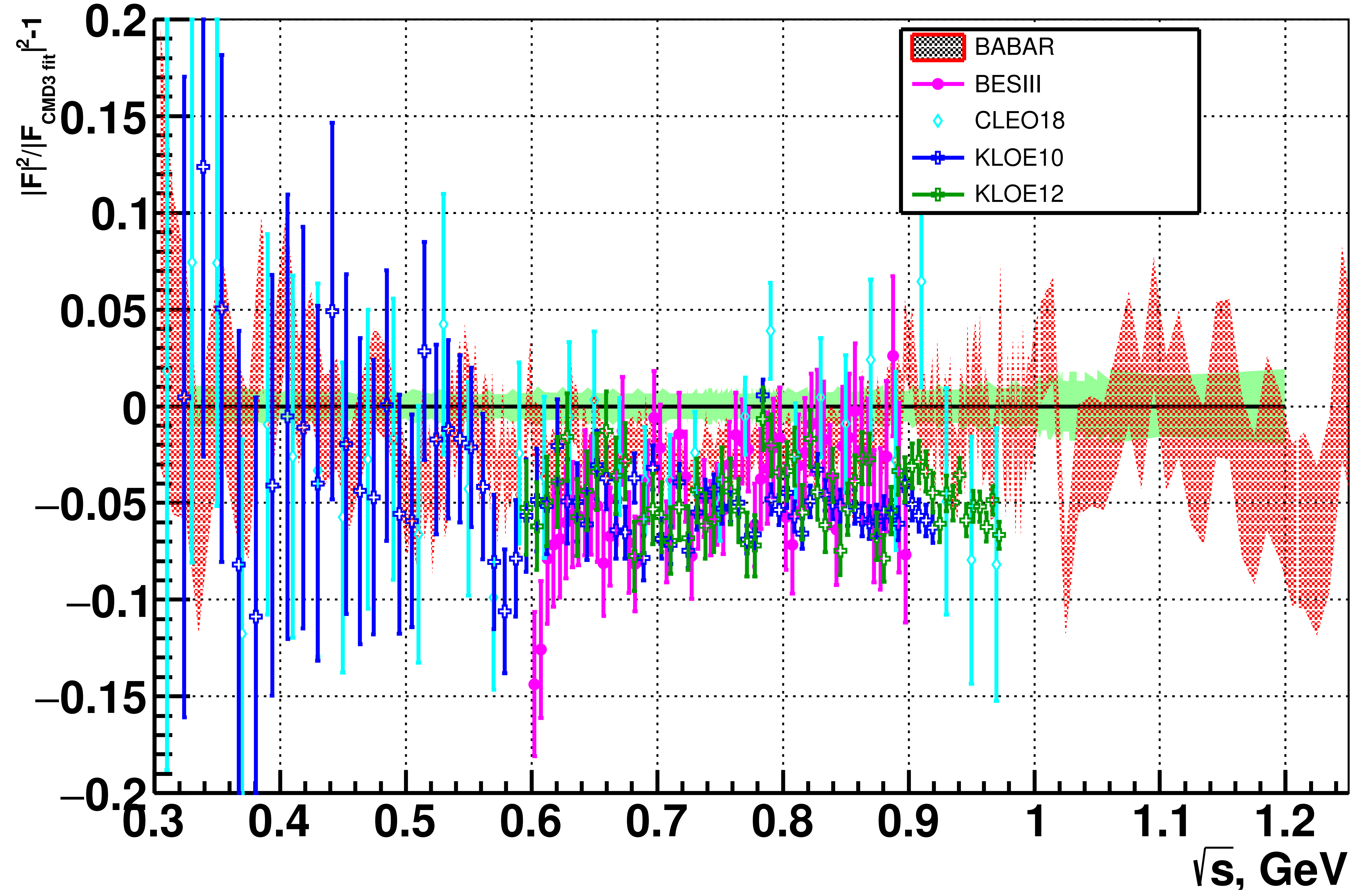


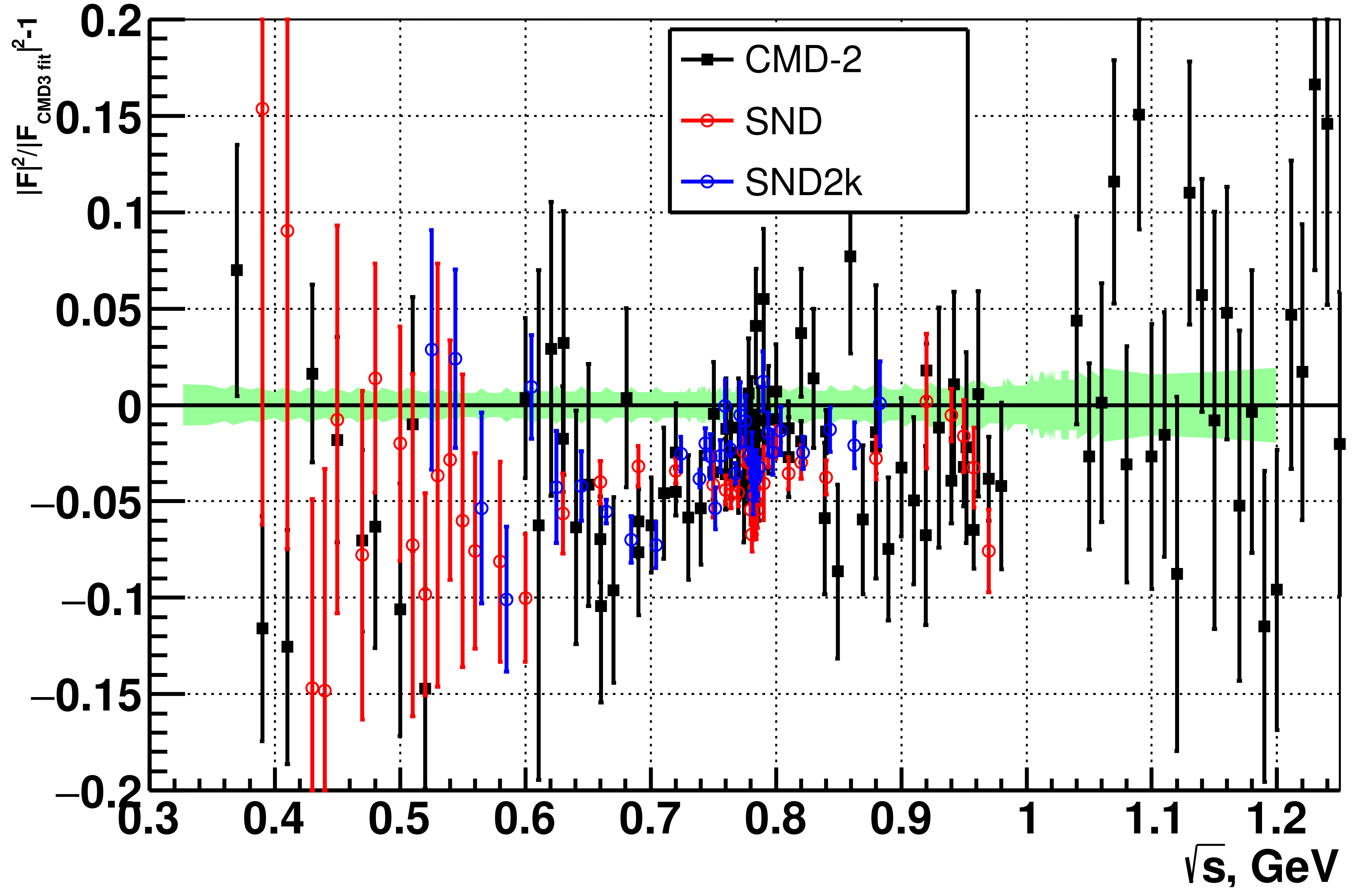
Laurent Lellouch @ Lattice @ CERN:

- Partial tail $a_{\mu,28\text{-}35}^{\text{LO-HVP}}$ for comparison with lattice dominated by cross section below ρ peak: $\sim 70\%$ for $\sqrt{s} \leq 0.63\,\text{GeV}$

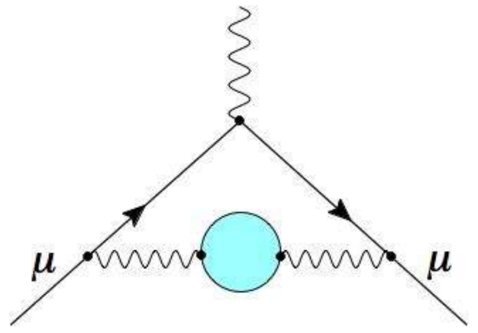
cross section comparisons

[CMD-3, F. Ignatov et al, arXiv:2302.08834, PRD2024]

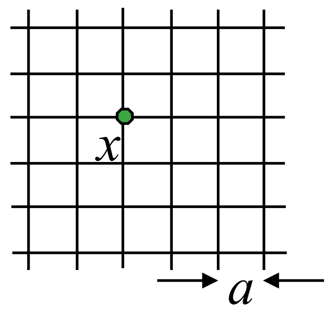




- For $\sqrt{s} \lesssim 0.6$ GeV: good consistency between cross section measurements
- For $0.6 \text{ GeV} \lesssim \sqrt{s} \lesssim 1$ GeV: significant differences between measurements



Lattice HVP: long-distance window

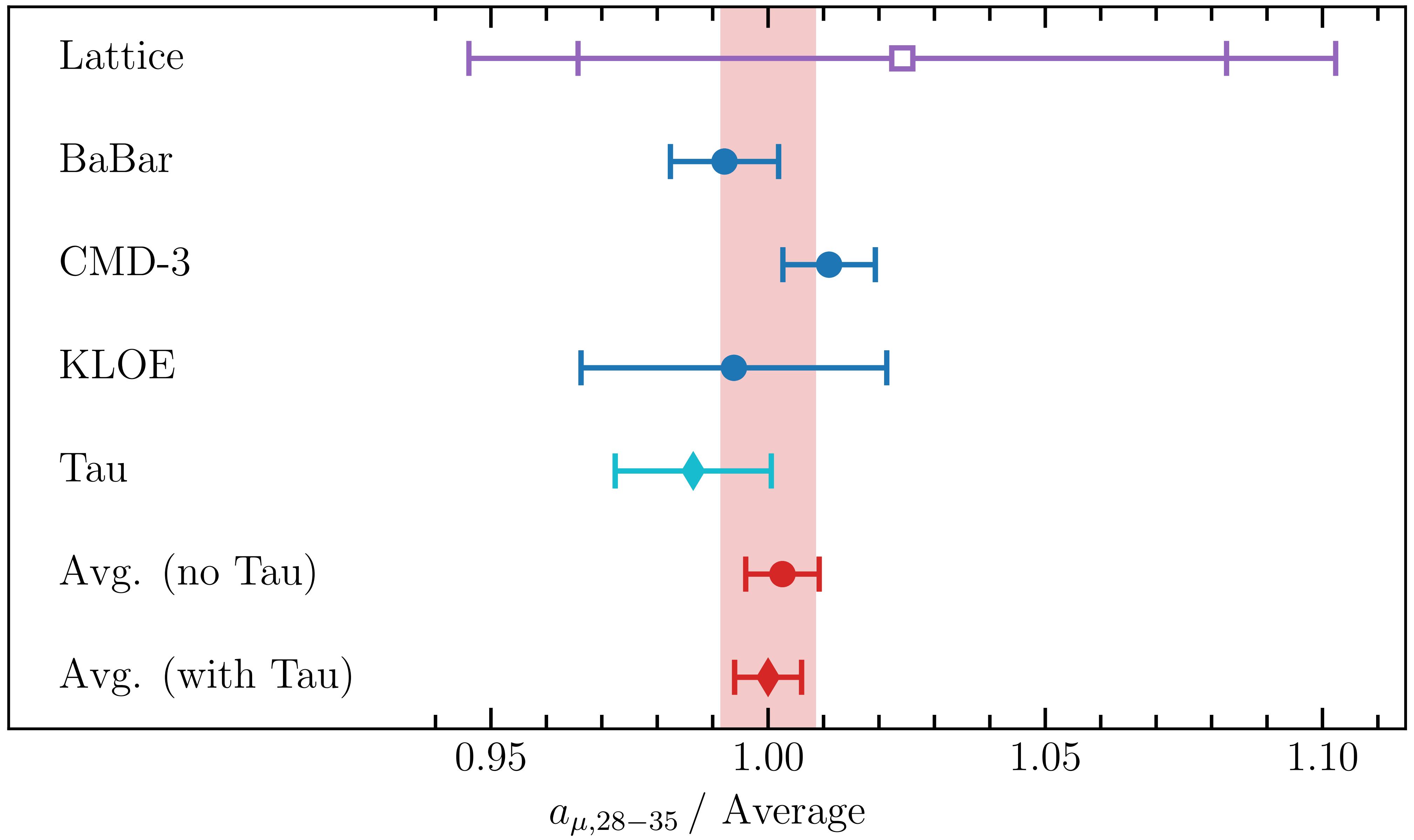


update: BMW+DMZ 2024

- BMW+DMZ 24 [A. Boccaletti et al, arXiv:2407.10913]

- statistical/systematic errors at long distances, $t \gtrsim 2.5$ fm still large:

[Laurent Lellouch Lattice @ CERN]



- All data-driven result agree very well
- Weighted average taken w/ and w/out τ: χ^2dof $= 1.1$ for both
- Final number: average w/ τ, PDG factor, and systematic $=$ full difference τ/no-τ added linearly

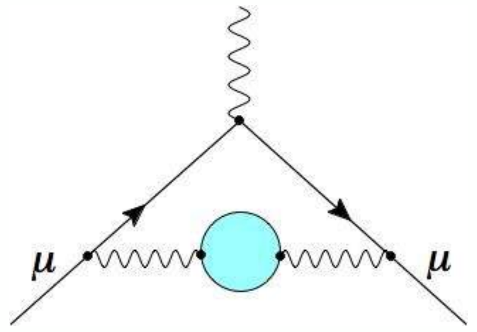
$$a_{\mu,28\text{-}35}^{\text{LO-HVP}} = 18.12(11)(5)[16]$$

- Excellent agreement w/ lattice, but uncertainty reduced by factor ~ 15
- Final number: average w/ τ, and systematic $=$ full difference τ/no-τ added linearly

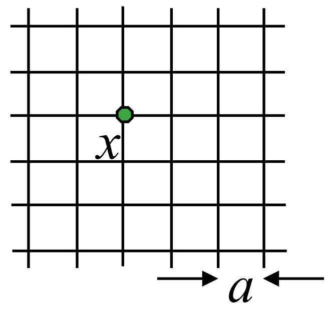
$$a_{\mu,28\text{-}\infty}^{\text{LO-HVP}} = 27.59(17)(9)[26]$$

- Only $\lesssim 5\%$ of final result for a_μ
- Contributes $\sim 65\%$ to total squared uncertainty uncertainty improvement: $5.5 \to 3.3$

➠ **hybrid evaluation:** combine lattice QCD calculation of one-sided window $a_\mu(t_1 = 2.8\text{ fm})$ with data-driven evaluation of long tail, $t > 2.8$ fm.



Lattice HVP: long-distance window

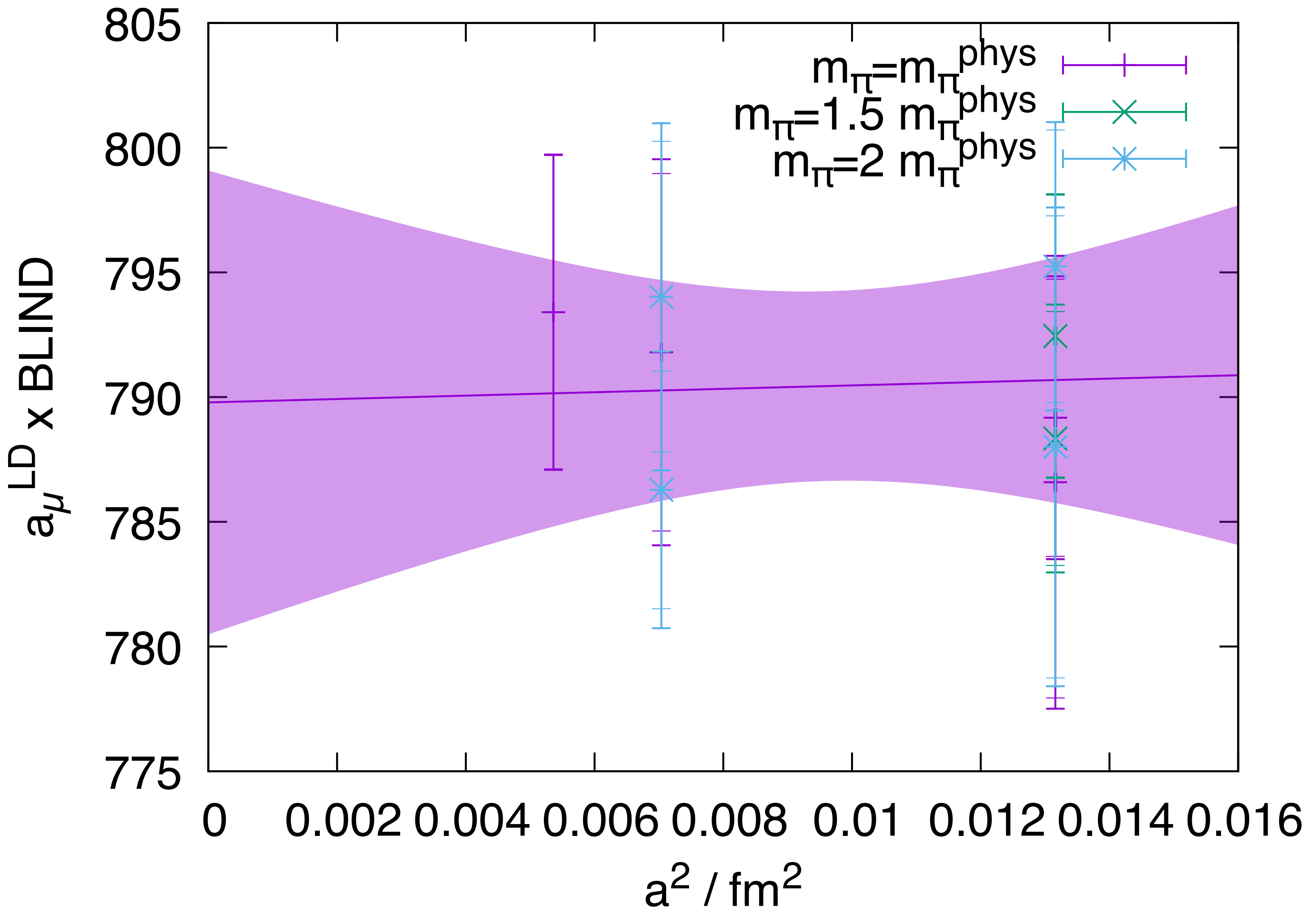


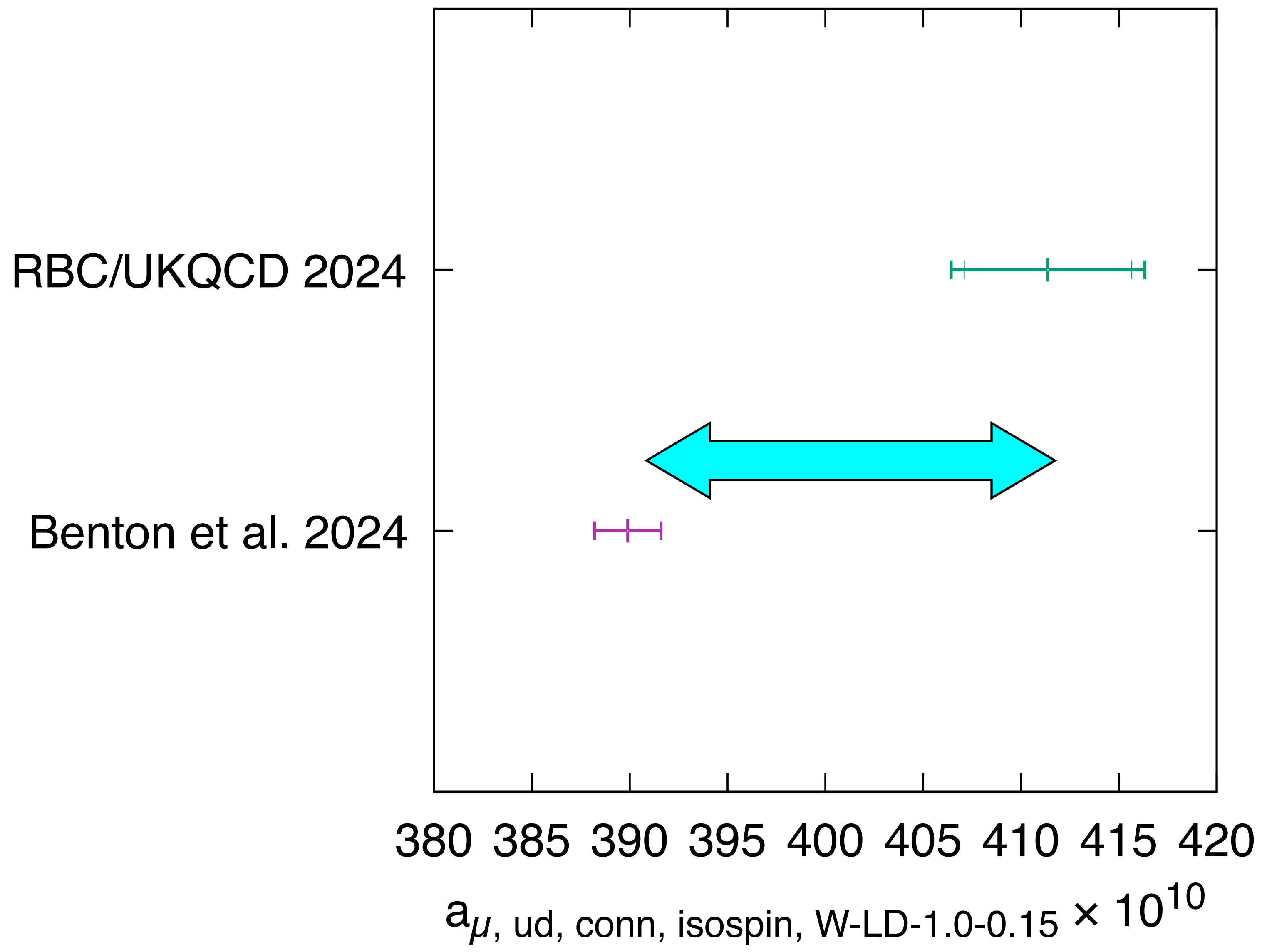
update: RBC/UKQCD 2024

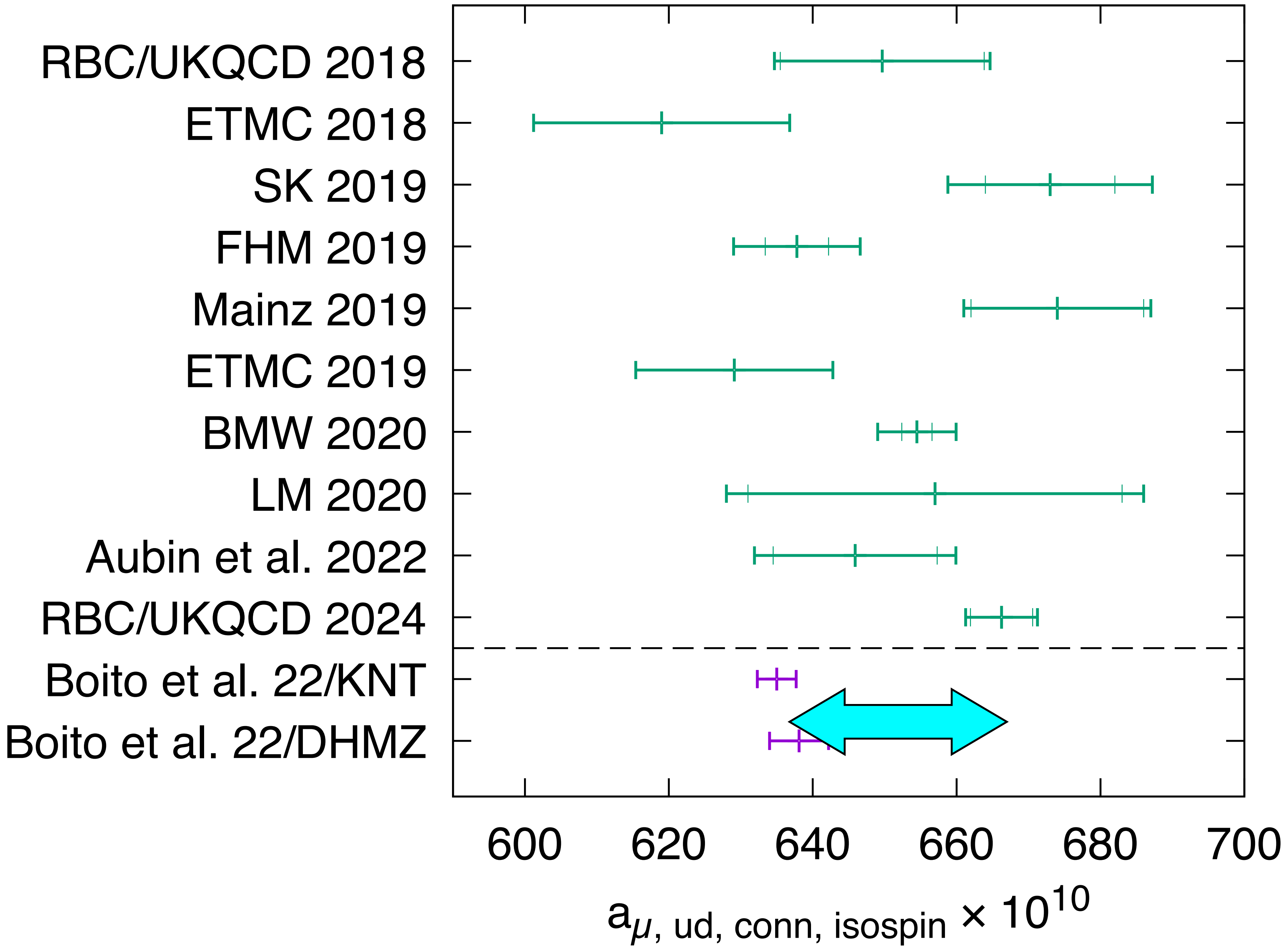
- long-distance window a_μ^{LD} and full a_μ^{ll}(conn.)

Unblinded results in BMW20 isospin-symmetric world

C. Lehner @ Lattice 2024



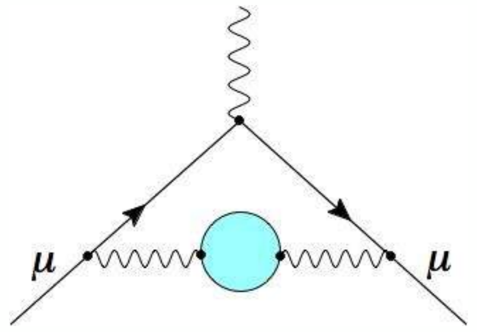




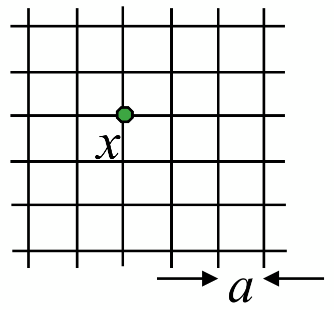
- "BMW20 world": fixed $w_0 = 0.1716$ fm scale uncertainty not included
- paper in progress

Result for $a_\mu^{\rm iso\ lqc}$ with 7.5/1000 precision.

$$a_\mu^{\rm LD\ iso\ lqc} = 411.4(4.3)_{\rm stat.}(2.3)_{\rm syst.} \times 10^{-10},$$
$$a_\mu^{\rm iso\ lqc} = 666.2(4.3)_{\rm stat.}(2.5)_{\rm syst.} \times 10^{-10}.$$



Lattice HVP: long-distance window

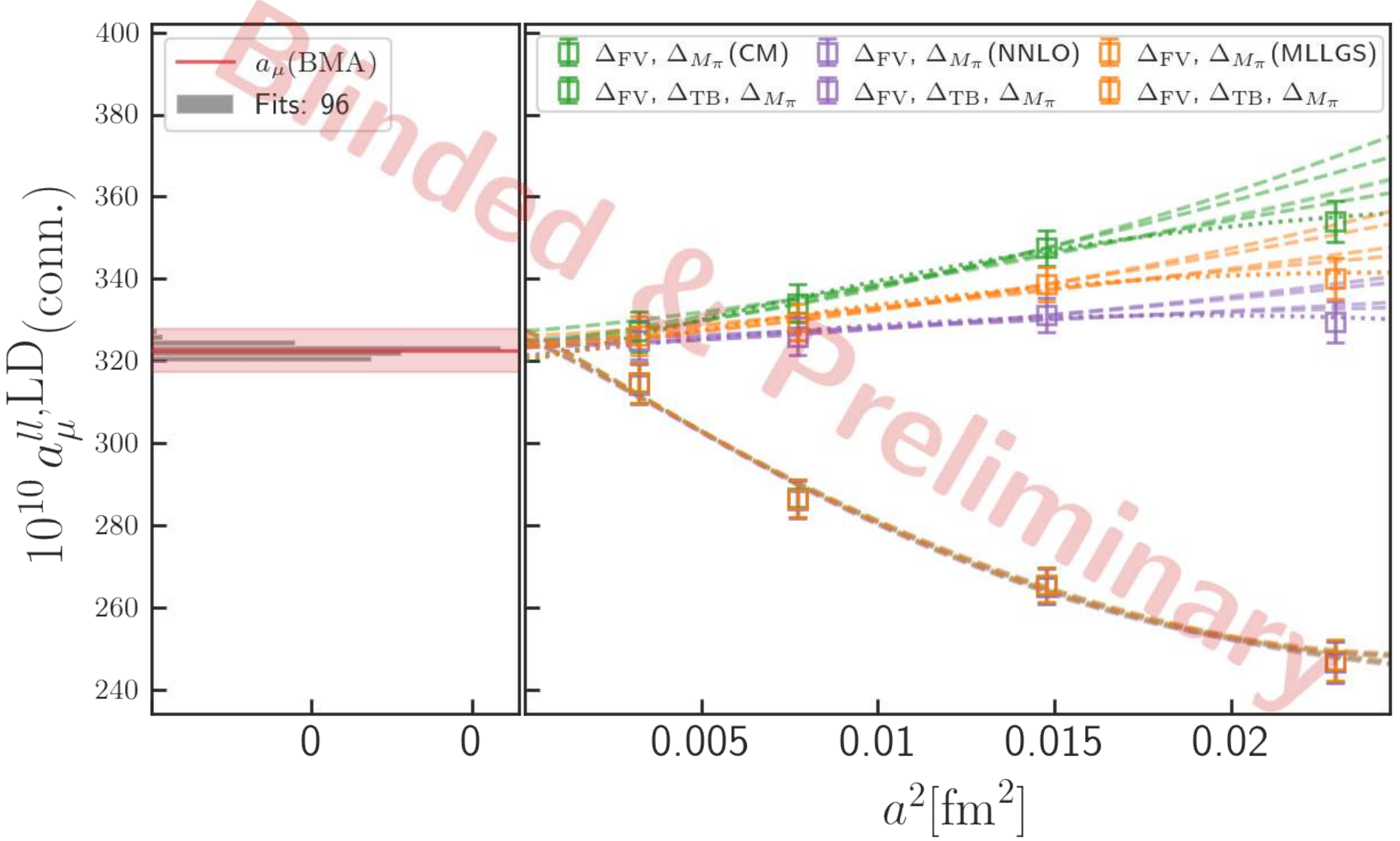


update: Fermilab/HPQCD/MILC 2024

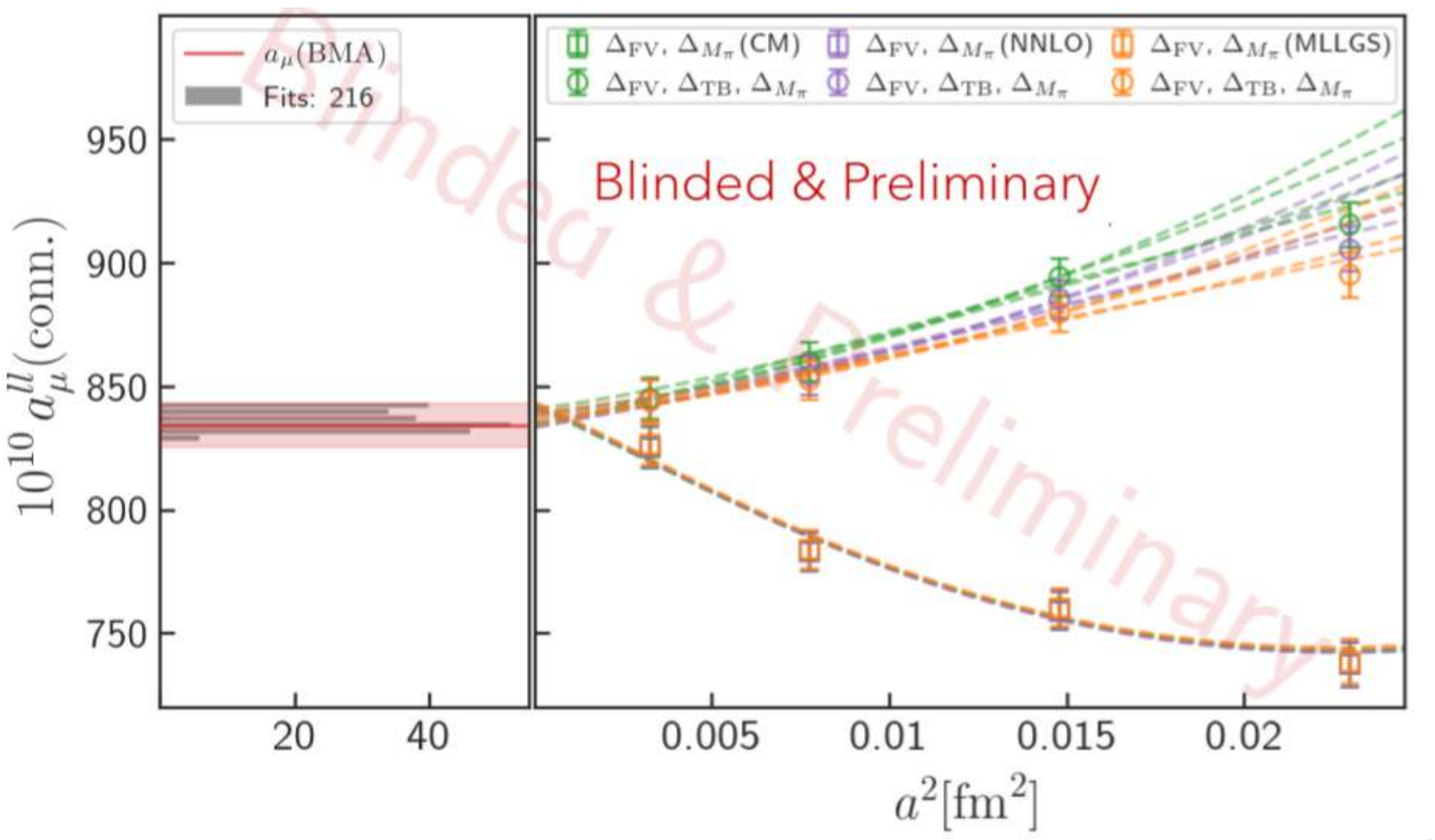
Michael Lynch @ Lattice 2024

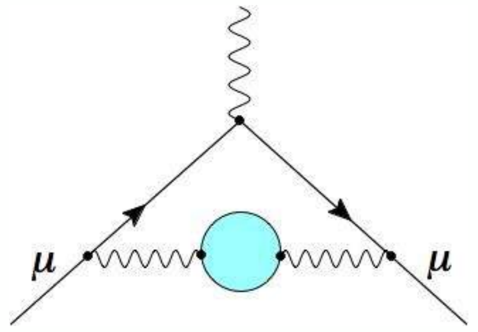
- long-distance window a_μ^{LD} and full $a_\mu^{ll}(\mathrm{conn.})$

$a_\mu^{ll}\,(\mathrm{conn.})$ - LD window

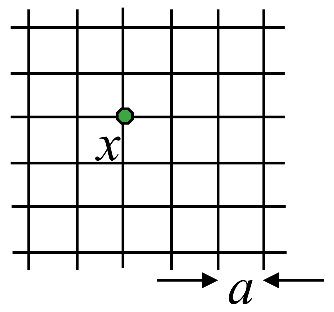


$a_\mu^{ll}\,(\mathrm{conn.})$ - Full





Lattice HVP: long-distance window

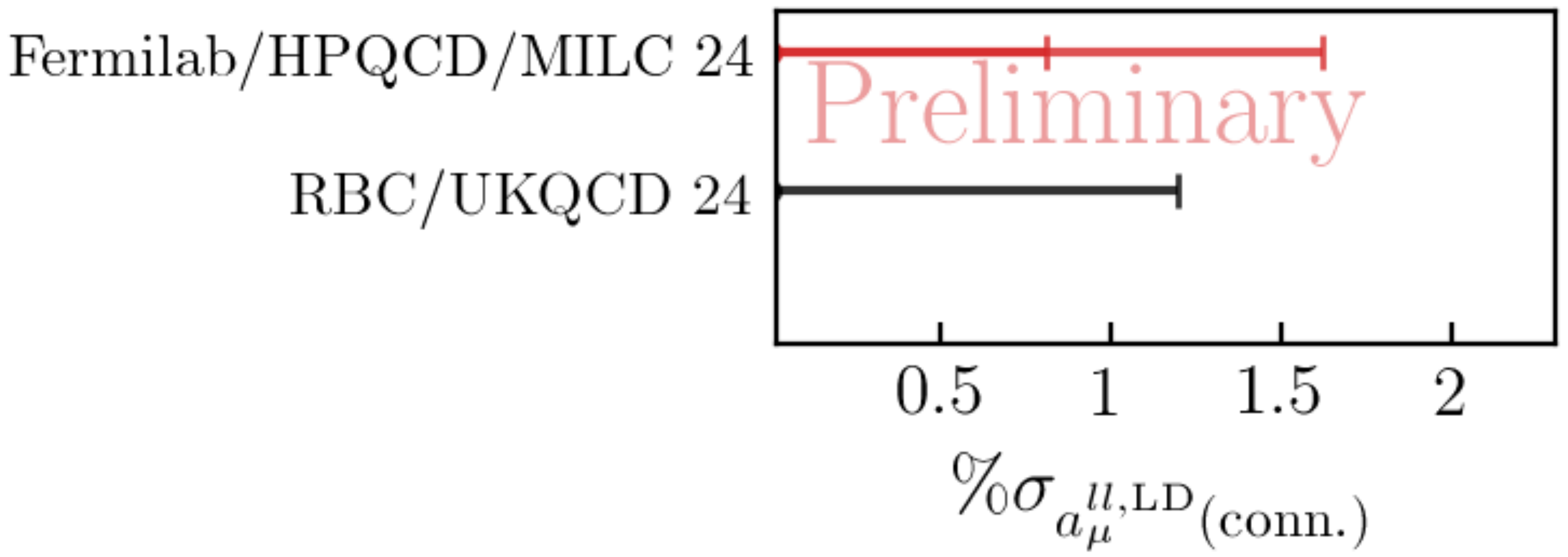


update: Fermilab/HPQCD/MILC 2024

Michael Lynch @ Lattice 2024

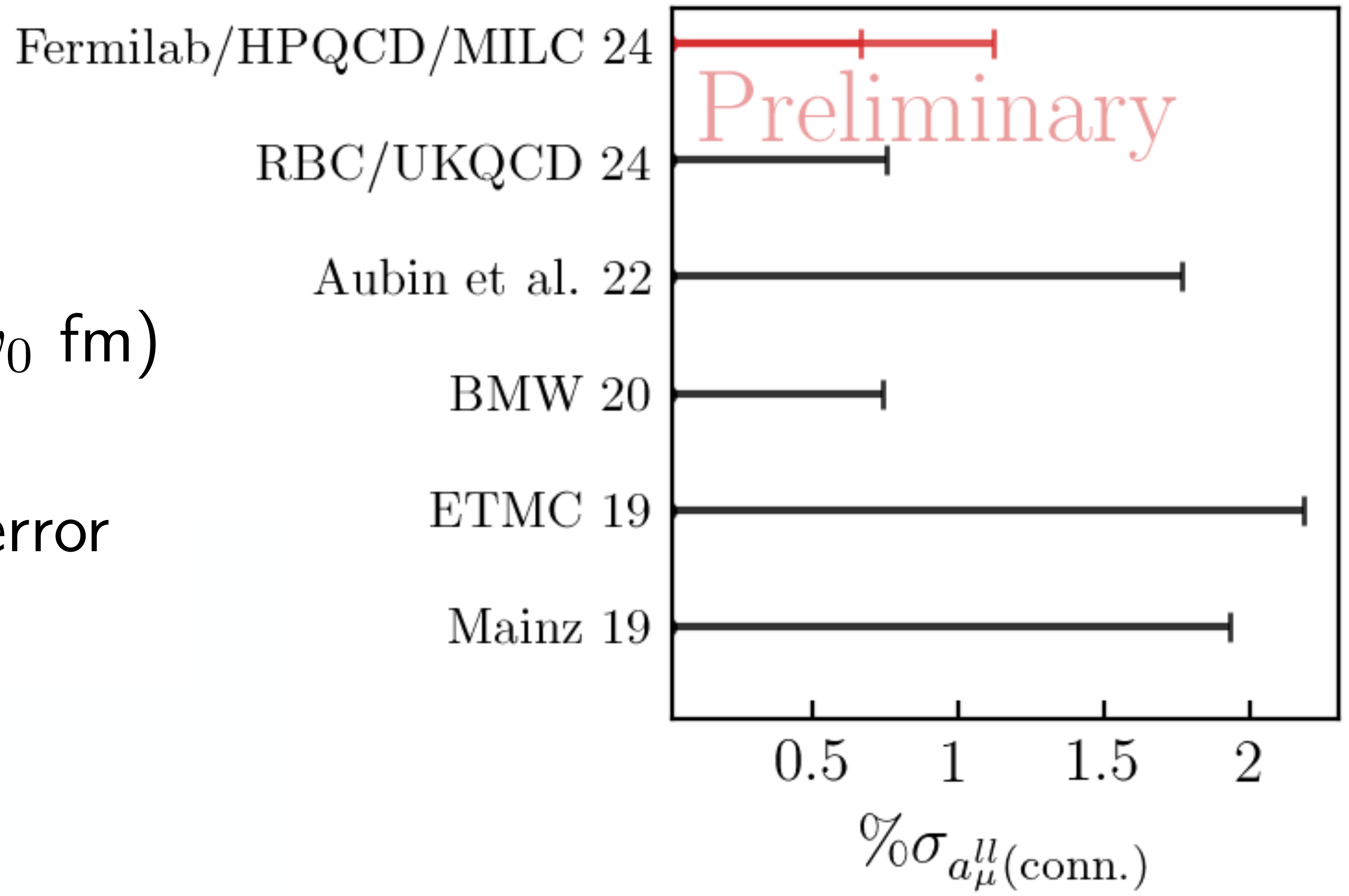
- long-distance window a_μ^{LD} and full $a_\mu^{ll}(\text{conn.})$

$a_\mu^{ll}\,(\text{conn.})$ - LD window



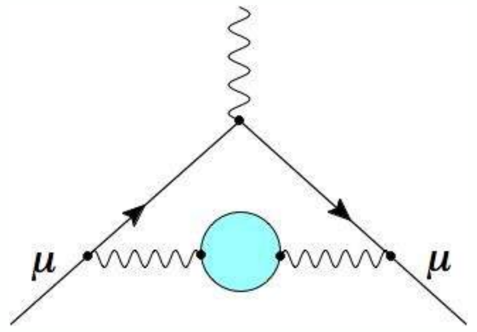
- Inner error w/o scale setting (w_0 fm) uncertainty
- Scale setting is now dominant error contributor.

$a_\mu^{ll}\,(\text{conn.})$ - Full

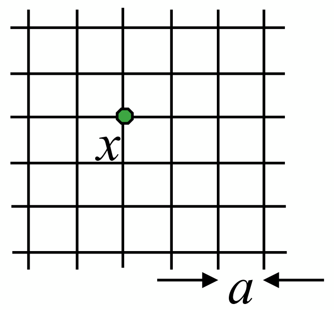


We expect further improvements in stat., sys. uncertainties from...

- Generation of correlator data at a lattice spacing of 0.04 fm is in progress.
- Improved scale setting via M_Ω.
- Joint fit analysis with multiple vector current discretizations
- Direct finite volume study: $L \sim 5.5$ fm $\to L \sim 11$ fm (at $a = 0.09$ fm) to replace EFT-based FV error estimates.
- Calculation of two-pion contributions to vector-current correlation functions at finer lattice spacings.



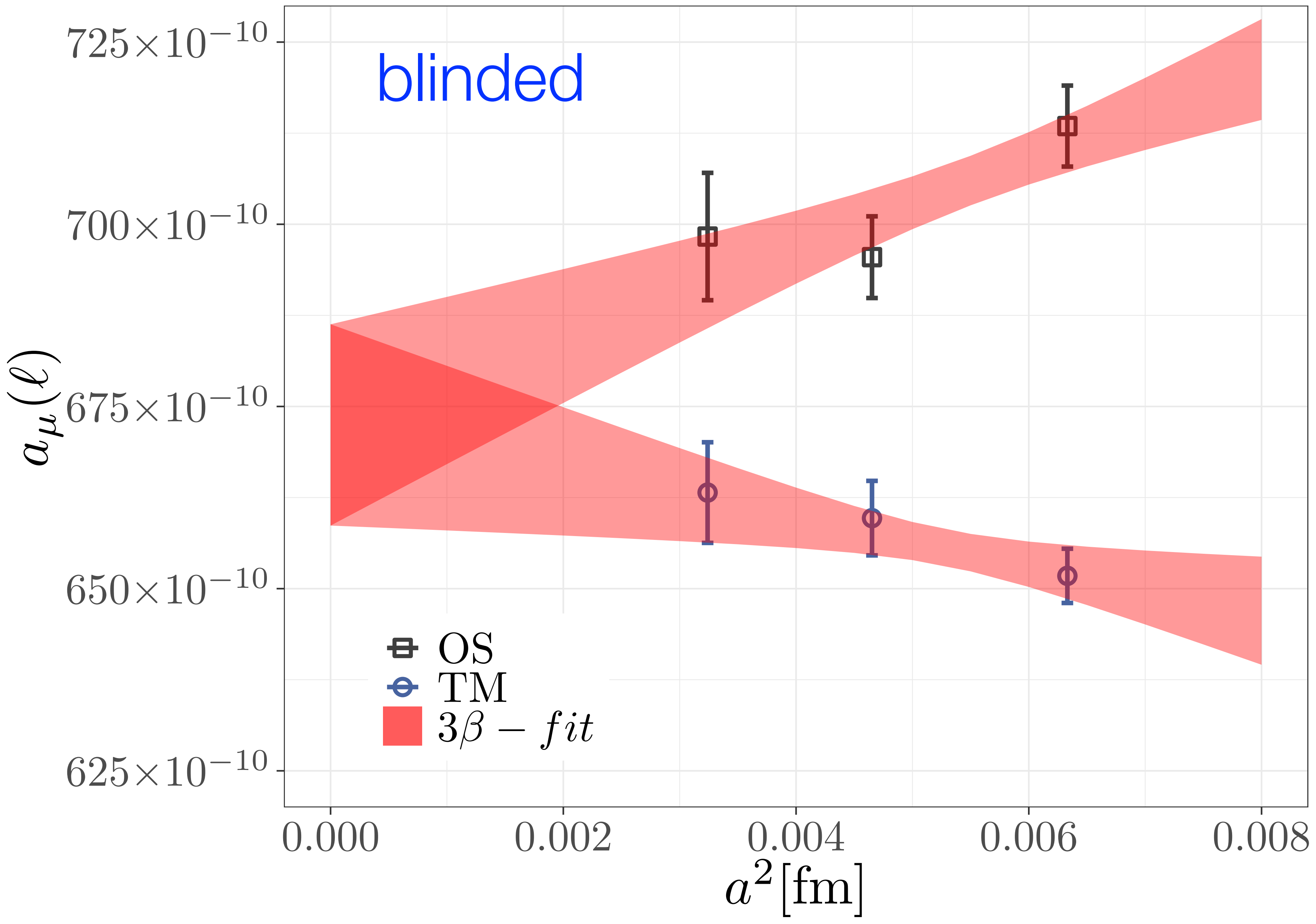
Lattice HVP: full window



update: ETMC 2024

M. Garofalo @ Lattice 2024

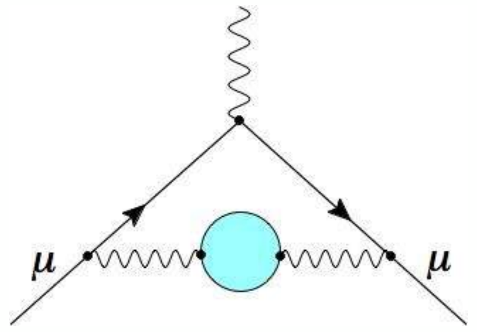
ℓ-quark connected [Preliminary]



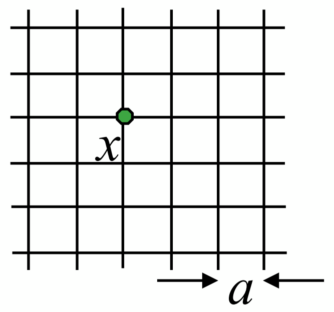
- Data extrapolated to $L = 5.46$ fm using Meyer-Lellouch-Lüscher-Gounaris-Sakurai (MLLGS) approach
- Correction to isoQCD pion point with MLLGS, lattice calculation in progress
- Significant reduction of uncertainty possible

* Statistic at the two finest lattice spacing not final, better bounding and a finer lattice spacing
* Considering models to reduce lattice artefacts and improve continuum extrapolation

also prelim. results for strange & charm



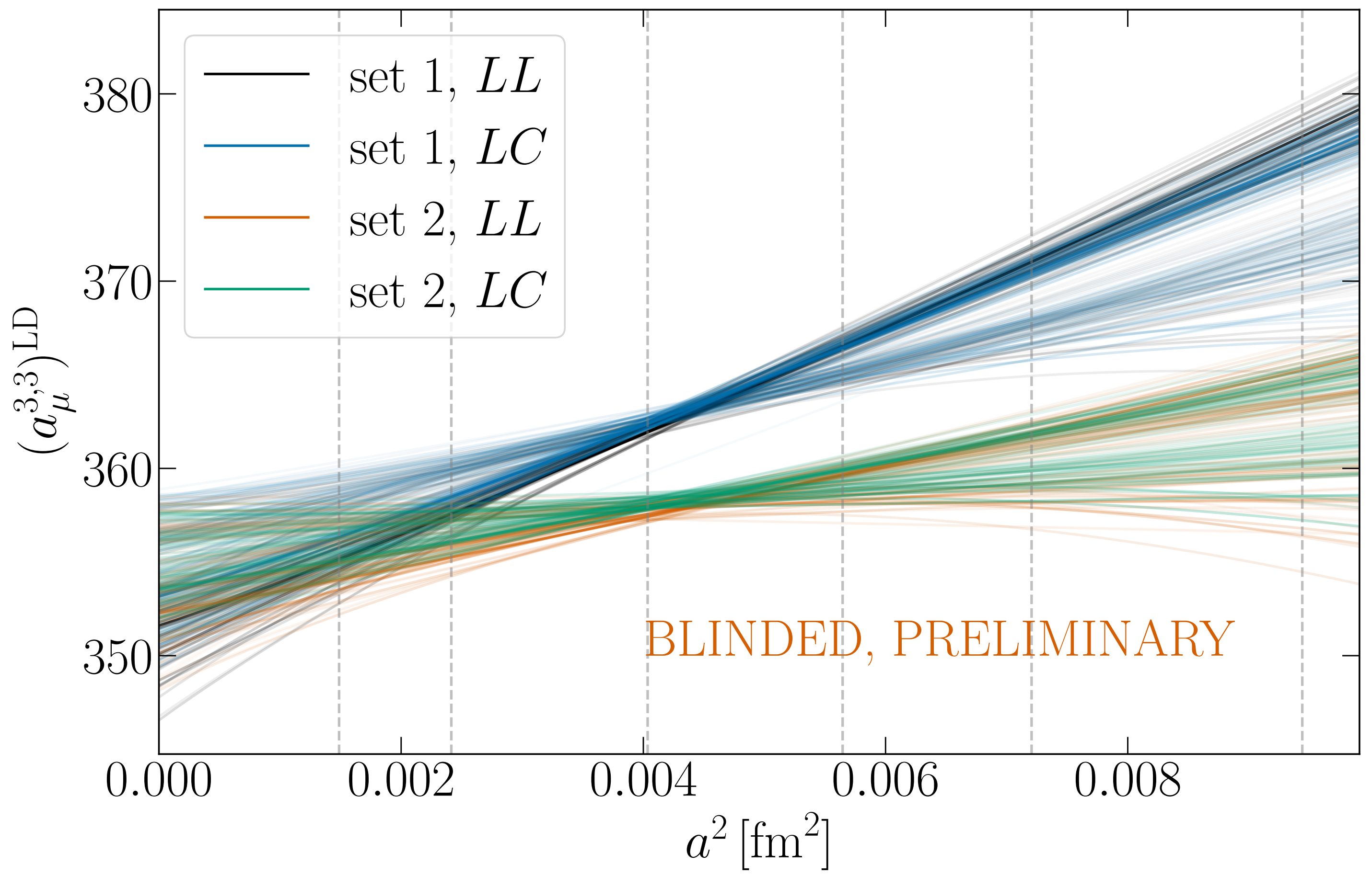
Lattice HVP: long-distance window



update: Mainz 2024

S. Kuberski @ Lattice 2024

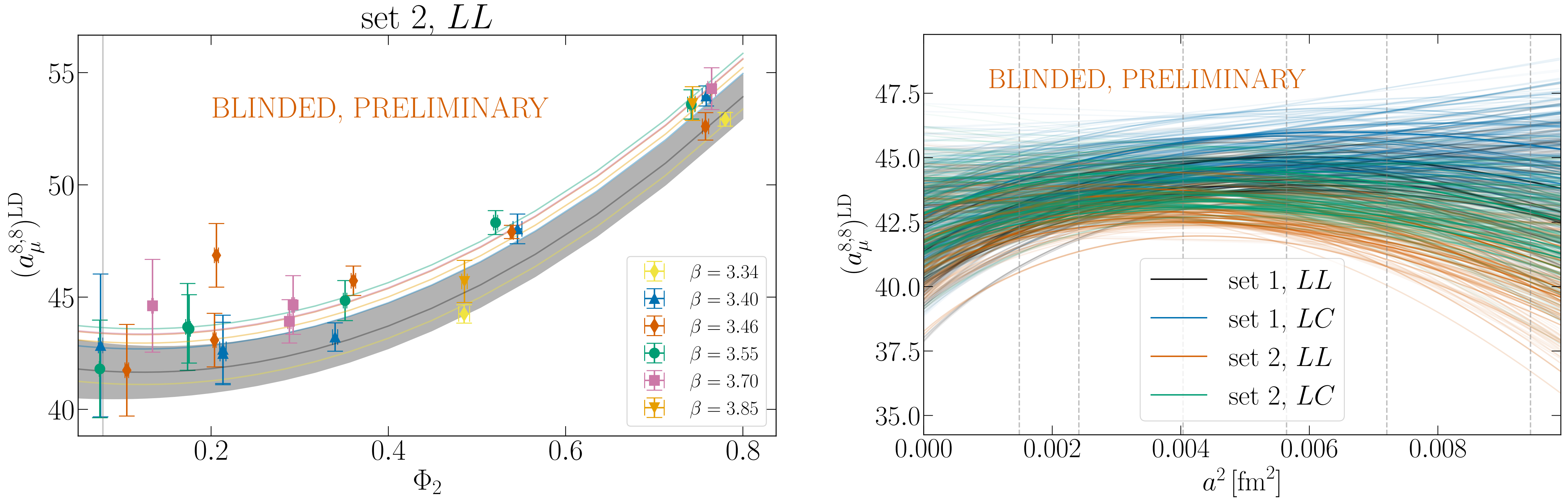
$(a_\mu^{\rm hvp})^{\rm LD}$ in the isovector channel: cutoff dependence



- Dependence of $(a_\mu^{3,3})^{\rm LD}$ on a^2 at physical quark masses.
- Four sets of data (colors) differ by $\mathrm{O}(a^2)$.
- Each line represents a fit in the model average.
- Include terms à la $[\alpha_s(1/a)]^{0.395}\, a^2$ [Husung, 2401.04303].

- Contains **artificial cutoff effects from the blinding** procedure.
- Higher order cutoff effects have a small weight in the model average.

$(a_\mu^{\rm hvp})^{\rm LD}$ in the isoscalar channel



- Quark-disconnected diagram contributes significantly to noise in the isoscalar channel, despite using multiple noise reduction techniques [Cè et al., 2203.08676].
- Bounding method in the isoscalar channel to tame the long-distance tail.
- Leading finite-size effects of light-connected and disconnected cancel.

$(a_\mu^{\rm hvp})^{\rm LD}$: Status and outlook

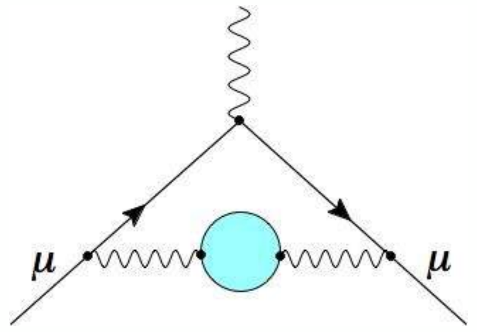
Achievements

- High statistical precision at $m_\pi^{\rm phys}$ and excellent control of the m_π dependence.
- Large span of lattice spacings to control the continuum extrapolation.

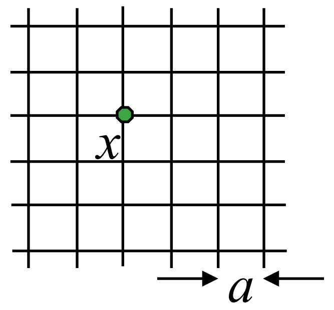
Challenges

- Scale setting remains a dominant source of uncertainty. The global status of gradient flow scales is unsatisfactory [FLAG23].
- Autocorrelation hinders precise estimates at very fine lattice spacing.
- Isospin breaking effects need to be computed accurately. [Julian Parrino, Thu 9:40] [Dominik Erb, Thu 10:00]

- Stay tuned for our unblinded result for $(a_\mu^{\rm hvp})^{\rm LD}$!



Lattice HVP: outlook



Ongoing work:

More windows:

- Use linear combinations of finer windows to locate the tension (if it persists) in $\sqrt{s}$
  [Colangelo et al, arXiv:12963]
- One-sided windows (excluding the long-distance region $t \gtrsim 2.5\,\mathrm{fm}$) to test data-driven evaluations [Davies at at, arXiv:2207.04765]

For total HVP and long-distance window:

- expect unblinded lattice results by fall from Mainz and FNAL/HPQCD/MILC
  ➠ check for consolidation
- Including $\pi\pi$ states for refined long-distance computation
  (Mainz, RBC/UKQCD, FNAL/MILC)
- smaller lattice spacings to test continuum extrapolations crucial

➠ Slides from Lattice 2024 conference

Summary

- consistent results from independent, precise LQCD calculations for light-quark connected contribution to **intermediate window** $a_\mu^{\rm W}$ (~ 1/3 of $a_\mu^{\rm HVP,LO}$) ➠ $>4\,\sigma$ tension with (pre-2023) data-driven results
- still need more independent LQCD results for long-distance contribution, total HVP: coming soon
  ➠ develop method average for lattice HVP results, **assess tensions (if any)**
- Programs and plans in place for:
  - data-driven HVP:
    new analyses from BaBar, KLOE, SND, Belle II,…. will shed light on current discrepancies
    (blind analyses are paramount!)
    improved treatment of structure dependent radiative corrections (NLO) in $\pi\pi$ and $\pi\pi\pi$ channels
  - lattice HVP: **if no tensions** between independent lattice results, $\sim 0.5\ \%$
  - dispersive HLbL and lattice HLbL

➠ H. Meyer, A. Kupsch, F. Hagelstein

- IF persistent tensions between data-driven and lattice HVP: will need detailed comparisons, explore connections between HVP, $\sigma(e^+e^-)$, $\Delta\alpha$, global EW fits.
- including τ decay data in data-driven approach:
  requires nonperturbative evaluation of IB correction [M. Bruno et al, arXiv:1811.00508]

➠ continued coordination by Theory Initiative: 2nd WP in progress

Outlook

m_{ud}, m_u/m_d, m_s

m_c, m_b

$\alpha_{\overline{MS}}(m_z)$

$\langle \bar{B}_q^0 | \mathcal{O}_i^{\Delta B=2} | B_q^0 \rangle$

$\langle \bar{D}^0 | \mathcal{O}_i^{\Delta C=2} | D^0 \rangle$

$\hat{B}_K, \ldots$

$f_{B_{(s)}}, \ldots$

$f^{K\to\pi}(q^2)$ $f_{+,0,T}^{B\to D}(q^2), \ldots$

$f_{K^\pm}$ $f_{+,0,T}^{B\to\pi}(q^2), \ldots$

$a_\mu^{\text{HVP LO}}$ a_μ^{HLbL}

$\Lambda_b \to (p, \Lambda_c, \Lambda)\, \ell\nu$

nucleon form factors, ..

g_A, g_T, g_S

$\langle \pi\pi_{(I=0)} | \mathcal{H}^{\Delta S=1} | K^0 \rangle$

$\langle \pi\pi_{(I=2)} | \mathcal{H}^{\Delta S=1} | K^0 \rangle$

d_n

PDFs, GPDs, TMDs,..

$D_s \to \ell\nu\gamma$

$K^+ \to \ell^+\nu\,(\gamma)$

$\pi\pi \to \rho$

ΔM_K, ϵ_K

MEs for light nuclei

$\langle NN | O_i | NN \rangle$

$B \to \pi\pi\, \ell\nu \ldots$

$B \to X_c \ell\nu$, other inclusive decay rates, ...

$\pi \to \ell^+\ell^-$, $K \to \gamma\gamma$

$K^+ \to \pi^+\ell^+\ell^-$ ...

$K^+ \to \pi^+\nu\bar{\nu}$

Complexity

LQCD flagship results

First complete LQCD results, large(ish) errors

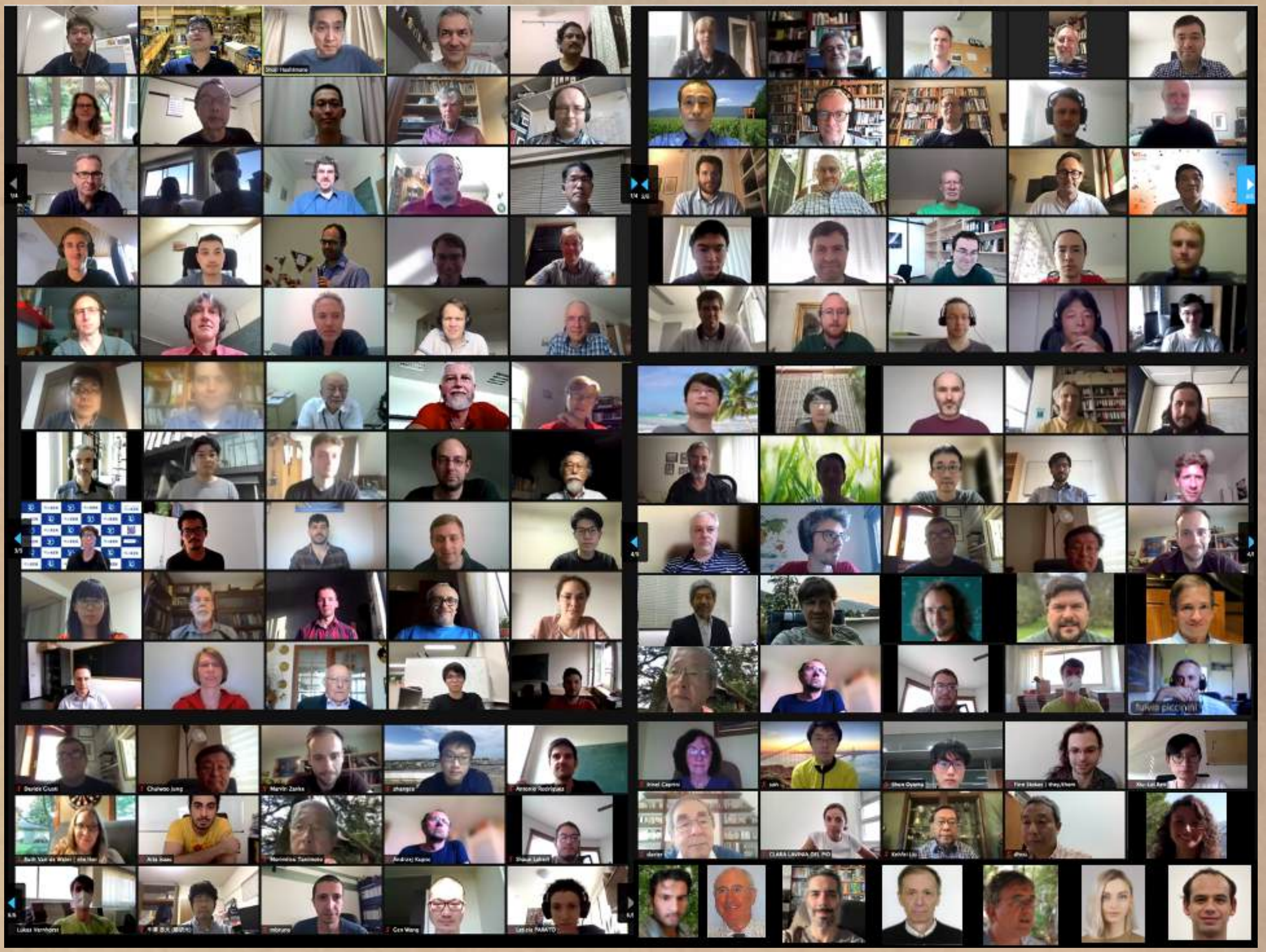
First results, physical params, incomplete systematics

new methods, pilot projects, unphysical kinematics

new ideas, first studies

Outlook

- Experimental program beyond 2025:
  - J-PARC: Muon g-2/EDM
  - CERN: MUonE
  - Fermilab: future muon campus experiments?
  - Belle II, BESIII, Novosibirsk,…
  - Chiral Belle (?)
- Data-driven/dispersive program beyond 2025:
  - development of NNLO MC generators
  - for HLbL, improved experimental/lattice inputs together with further development of dispersive approach
- MUonE will provide a space-like determination of HVP
- Lattice QCD beyond 2025:
  - access to future computational resources (coming Exascale) will enable improvements of all errors (statistical and systematic)
  - concurrent development of better methods and algorithms (gauge-field sampling, noise reduction) will accelerate progress
  - **beyond g-2: a rich program relevant for all areas of HEP**



CENTRE FOR THEORETICAL PHYSICS Higgs

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Thank you!

epcc

URA UNIVERSITIES RESEARCH ASSOCIATION

STRONG 2020

Fermi Research Alliance LLC

HIM HELMHOLTZ Helmholtz-Institut Mainz

SFB 1044 THE LOW-ENERGY FRONTIER OF THE STANDARD MODEL

PRISMA

Ip3

UCONN DEPARTMENT OF PHYSICS

JGU JOHANNES GUTENBERG UNIVERSITÄT MAINZ

7th Plenary Workshop of the Muon g−2 Theory Initiative

September 9-13, 2024 @ KEK, Tsukuba, Japan

https://conference-indico.kek.jp/event/257

g-2

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$(g\text{-}2)_7$

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Appendix

Bayesian Model Averaging

Ethan Neil @ Lattice 2024

➠ see also poster by Jake Sitison

- *Some history*: we didn't bring model averaging to lattice, we "added the B" (**Bayesian** MA), found new ICs, and tried to clarify statistical derivations/details.
- Several early variations of model averaging/ variation appear in lattice papers: Y. Chen et al. '04, **BMW '08**, **HPQCD '08**, **FNAL/MILC '14**, BMW '14…however, many old papers use *ad hoc* averaging prescriptions.
- First use of AIC for lattice is BMW '15; see also **CalLat '18**, '20, Rinaldi et al. '19. (More refs in our paper, including statistics papers back to the '70s.)
- First use of AIC with data penalty is BMW '21 (although I will argue for a *corrected* version of their formula here.)

[Y. Chen et al '04]: arXiv:**hep-lat/0405001**
[BMW '14]: PRD 90 (2014), arXiv:**1310.3626**
[BMW '15]: Science 347 (2015), arXiv:**1406.4088**
[Rinaldi et al. '19]: PRD 99 (2019), arXiv:**1901.07519**
[CalLat '20]: PRD 102 (2020), arXiv:**2005.04795**
[BMW '21]: Nature 593 (2021), arXiv:**2002.12347**

- Bayesian model averaging: obtain any expectation value as a weighted average

$$\langle O \rangle = \sum_M \langle O \rangle_M \ \mathrm{pr}(M|D)$$

- Note that this applies at the level of *expectation values*. In particular, for mean and variance we find:

$$\langle f(\mathbf{a}) \rangle = \sum_\mu f(\mathbf{a}^*_\mu) \mathrm{pr}(M_\mu|\{y\}),$$

$$\sigma^2_{f(\mathbf{a})} = \langle f(\mathbf{a})^2 \rangle - \langle f(\mathbf{a}) \rangle^2$$

$$= \sum_\mu \sigma^2_{f(\mathbf{a}_\mu)} \mathrm{pr}(M_\mu|\{y\}) + \sum_\mu f(\mathbf{a}^*_\mu)^2 \mathrm{pr}(M_\mu|\{y\}) - \left(\sum_\mu f(\mathbf{a}^*_\mu) \mathrm{pr}(M_\mu|\{y\}) \right)^2 ,$$

average stat. error — model-variation systematic

- Asymptotically correct model weights are given by the (Bayesian) Akaike information criterion (AIC):

$$-2 \log \mathrm{pr}(M|D) = -2 \log \mathrm{pr}(M) + \mathrm{BAIC}$$

$$\mathrm{BAIC} = \hat{\chi}^2(\mathbf{a}^*) + 2k$$

pr(M) is model prior prob - unless you know what this is, take it to be uniform and ignore it.

- This is *not* the same as taking a weighted average of variances (first term), *or* taking the variance of the weighted f(a*).

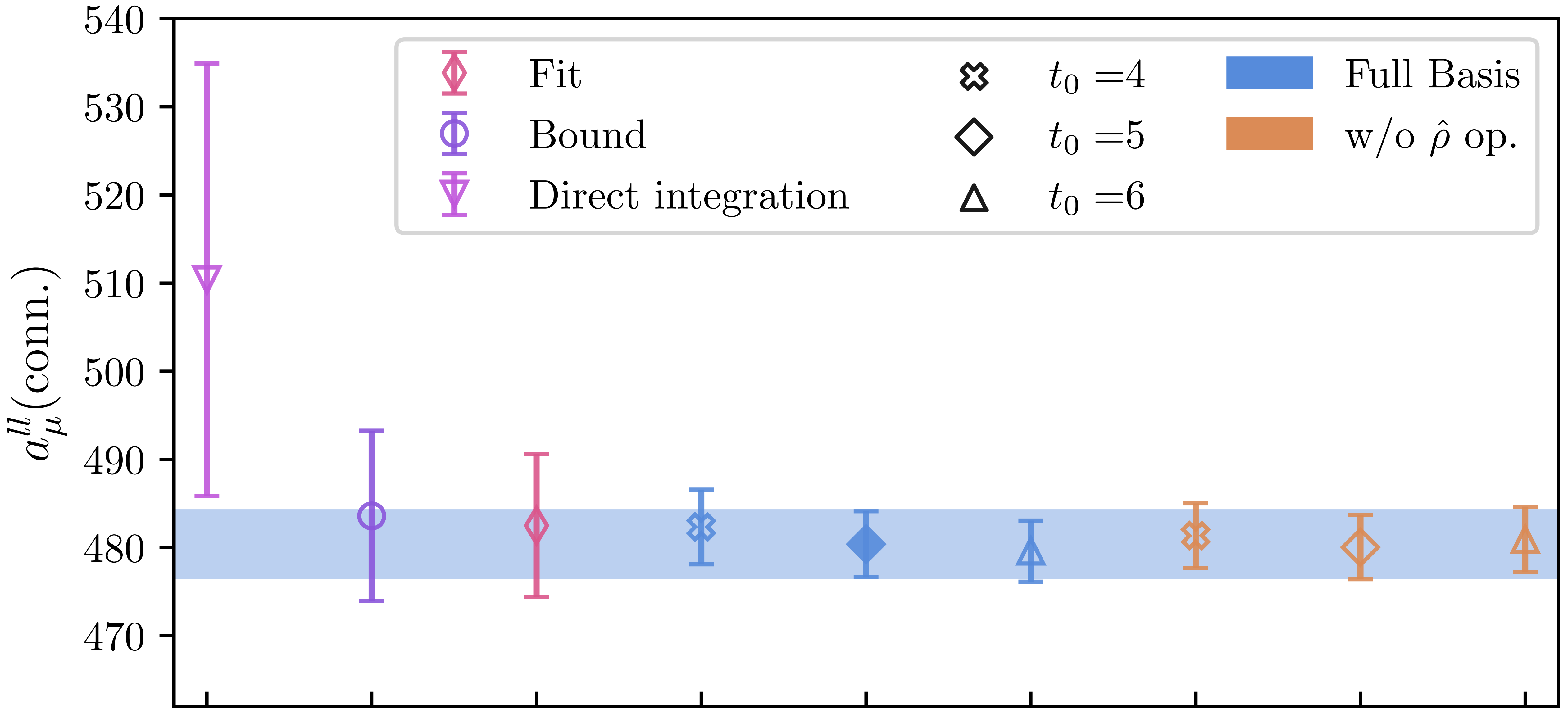
Long-distance tail (ud)

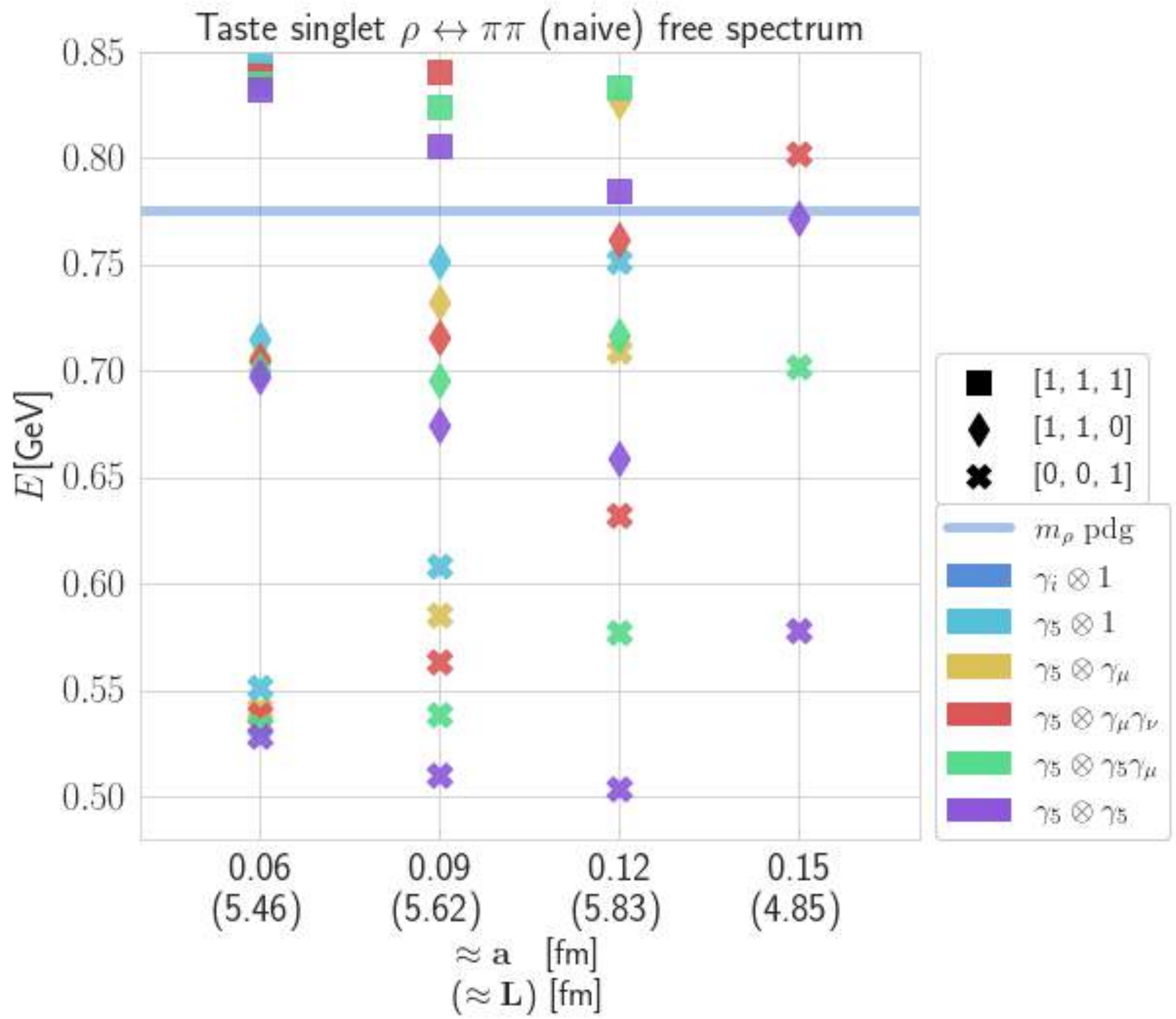
[Shaun Lahert et al, arXiv: 2409.00756]

$$C(t) = \frac{1}{3}\sum_{i,x}\langle j_i^{\mathrm{EM}}(x,t)\, j_i^{\mathrm{EM}}(0,0)\rangle$$

- Spectral reconstruction: $C(t) = \sum_{n=0}^{\infty} |A_n|^2 e^{-E_n t}$
  - ✦ obtain low-lying finite-volume spectrum (E_n, A_n) in dedicated study using additional operators that couple to two-pion states
  - ✦ First LQCD calculation with staggered multi-pion operators
  - ✦ Construct matrix of correlators (2,3,4-point functions)
  - ✦ Use GEVP to obtain energies and amplitudes for $\pi\pi$ states
  - ✦ Reconstruct vector-current correlator

$$\mathbf{C}(t) = \begin{pmatrix} C(t)_{J,\tilde{J}\to J,\tilde{J}} & C(t)_{J,\tilde{J}\to\pi\pi} \\ C(t)_{\pi\pi\to J,\tilde{J}} & C(t)_{\pi\pi\to\pi\pi} \end{pmatrix}$$

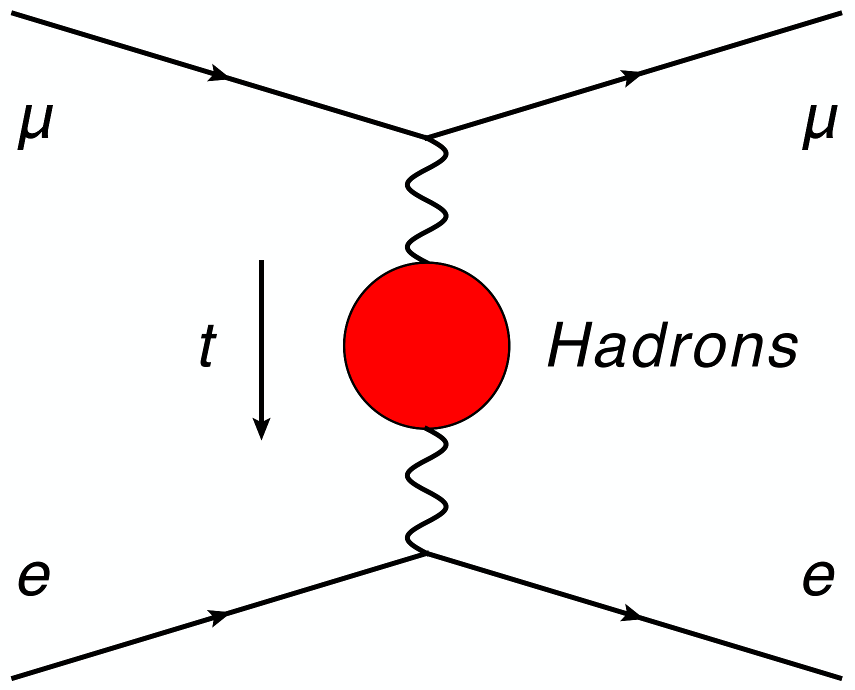


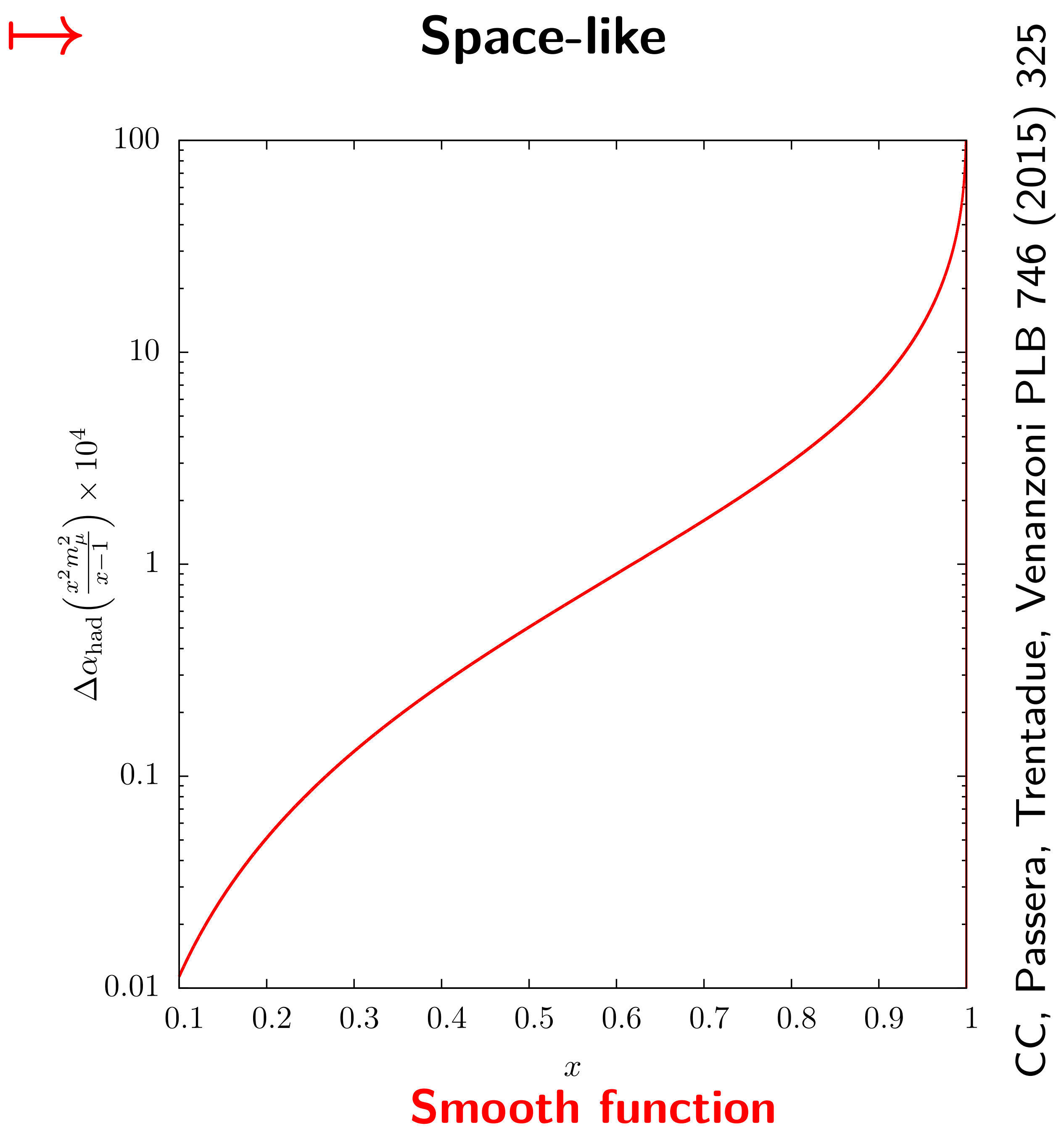


Hadronic vacuum polarization

μ-e elastic scattering to measure $a_\mu^{\rm HVP}$

LOI June 2019 [P. Banerjeei et al, arXiv:2004.13663, Eur.Phys.J.C 80 (2020)]





- use CERN M2 muon beam (160 GeV)
- Physics beyond colliders program @ CERN
- LOI June 2019
- pilot run in 2022, adding more stations 2023-2026
- goal < 0.5% uncertainty

Theoretical progress:
development of $\mu - e$ (N)NLO MC and computation of QED corrections

Connections

$$\sigma(e^+e^- \to \text{hadrons}) \quad \Leftrightarrow \quad a_\mu^{\text{HVP}} \quad \Leftrightarrow \quad \Delta\alpha_{\text{had}}(M_Z^2)$$

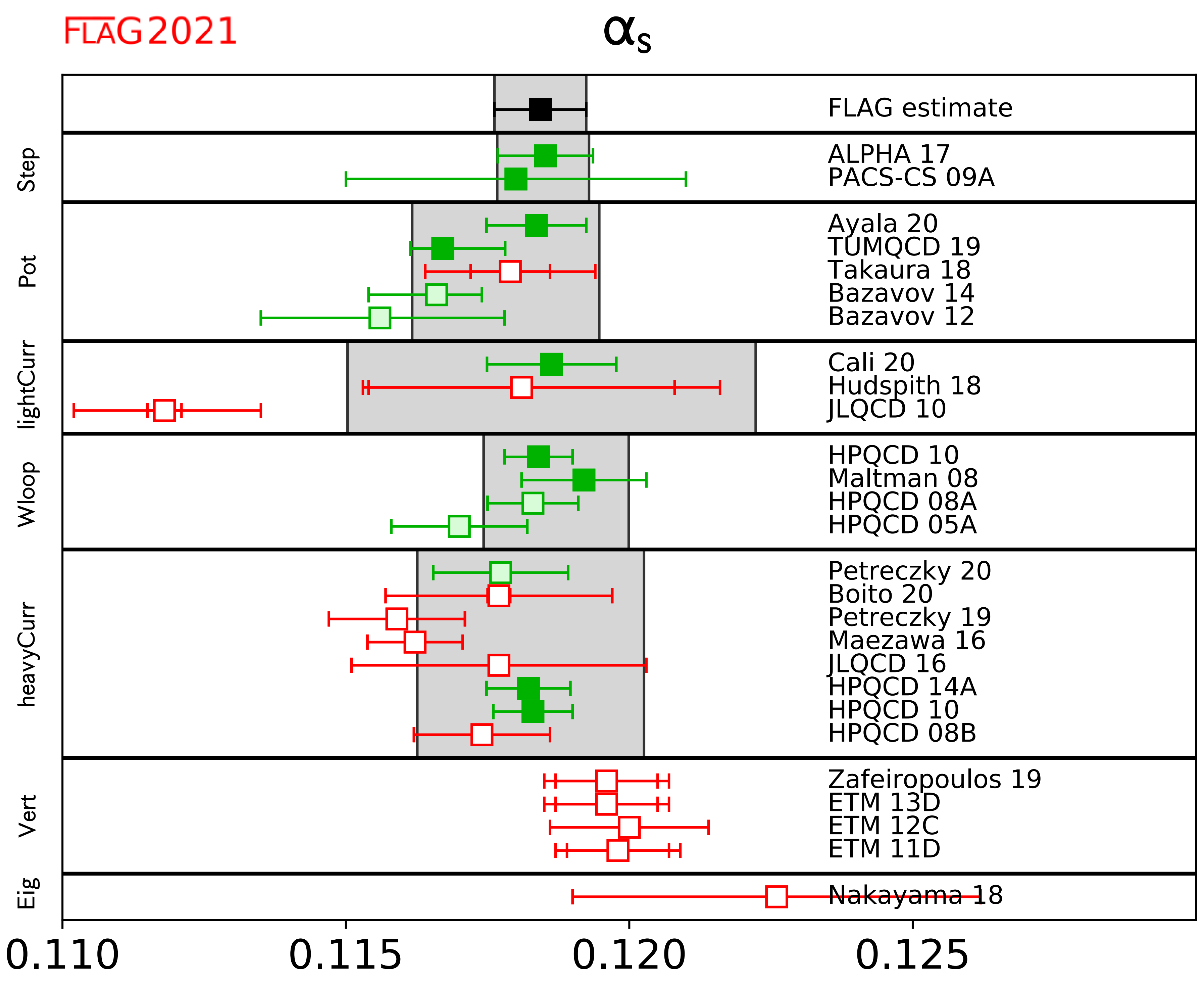
- $\Delta\alpha_{\text{had}}(M_Z^2)$ also depends on the hadronic vacuum polarization function, and can be written as an integral over $\sigma(e^+e^- \to \text{hadrons})$, but weighted towards higher energies.
- a shift in a_μ^{HVP} also changes $\Delta\alpha_{\text{had}}(M_Z^2)$: ➠ EW fits

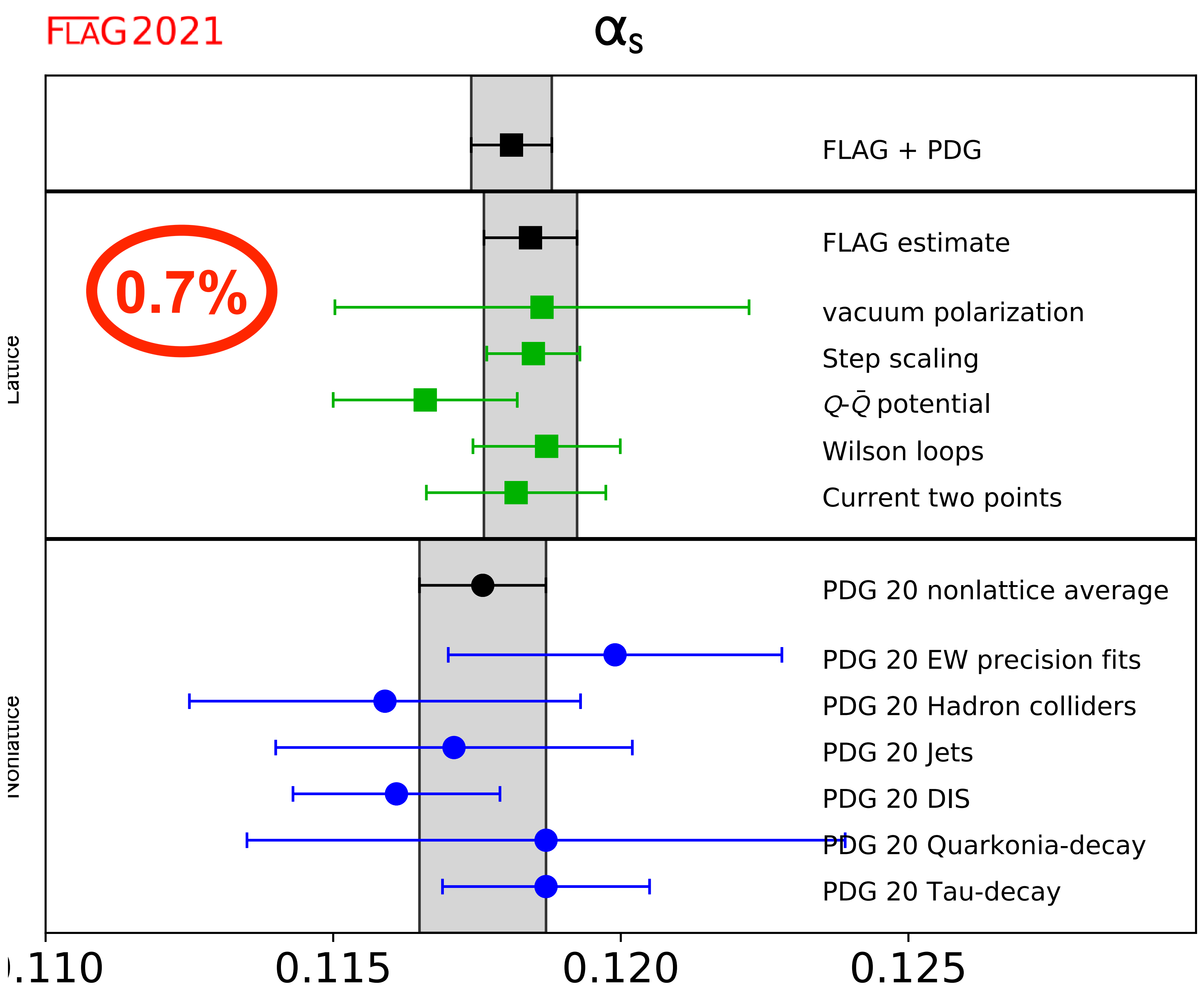
  [Passera, et al, 2008, Crivellin et al 2020, Keshavarsi et al 2020, Malaescu & Scott 2020]

  If the shift in a_μ^{HVP} is in the low-energy region ($\lesssim 1\,\text{GeV}$), the impact on $\Delta\alpha_{\text{had}}(M_Z^2)$ and EW fits is small.

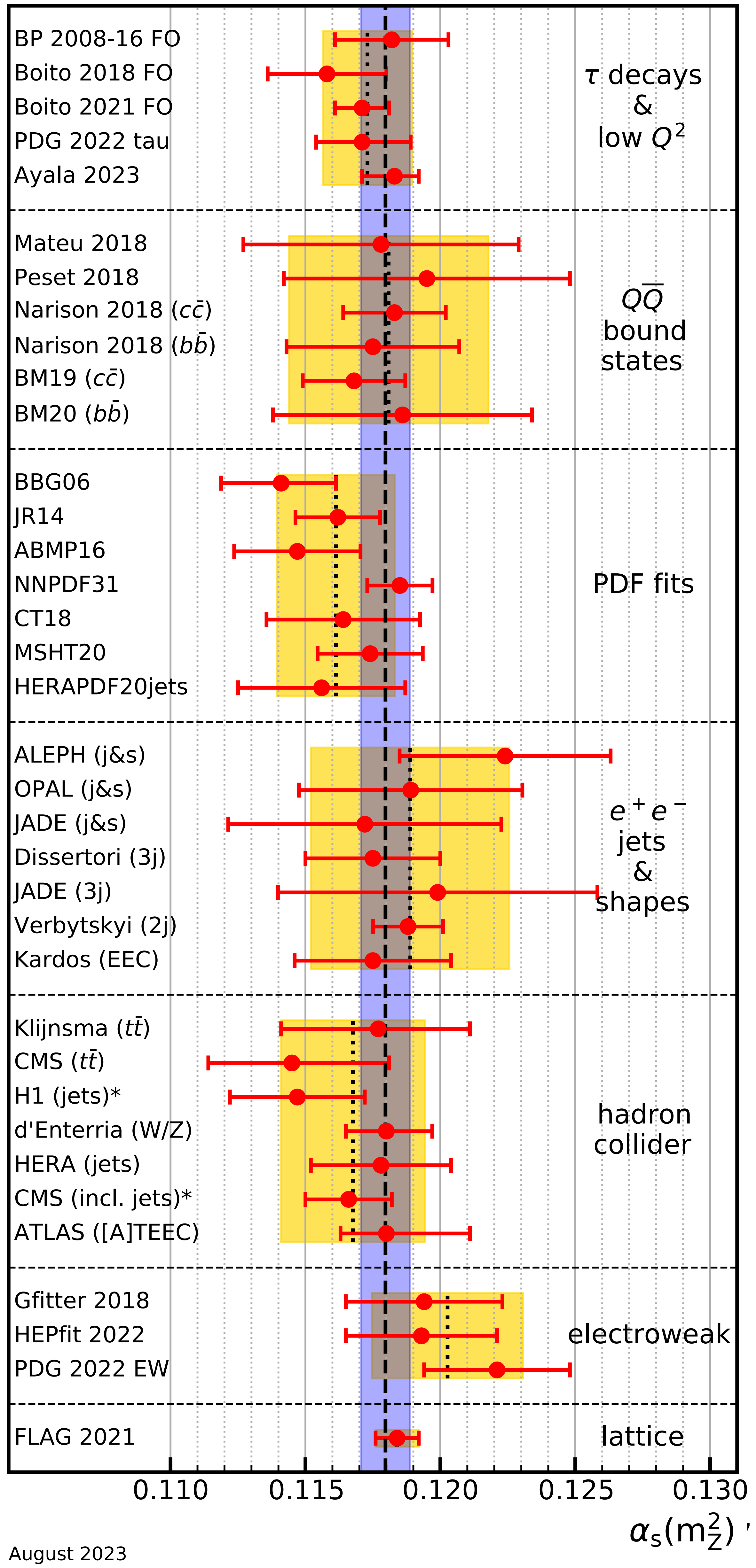
α_s

S. Aoki et al [FLAG 2021 review, arXiv:2111.09849, EPJC 2022]

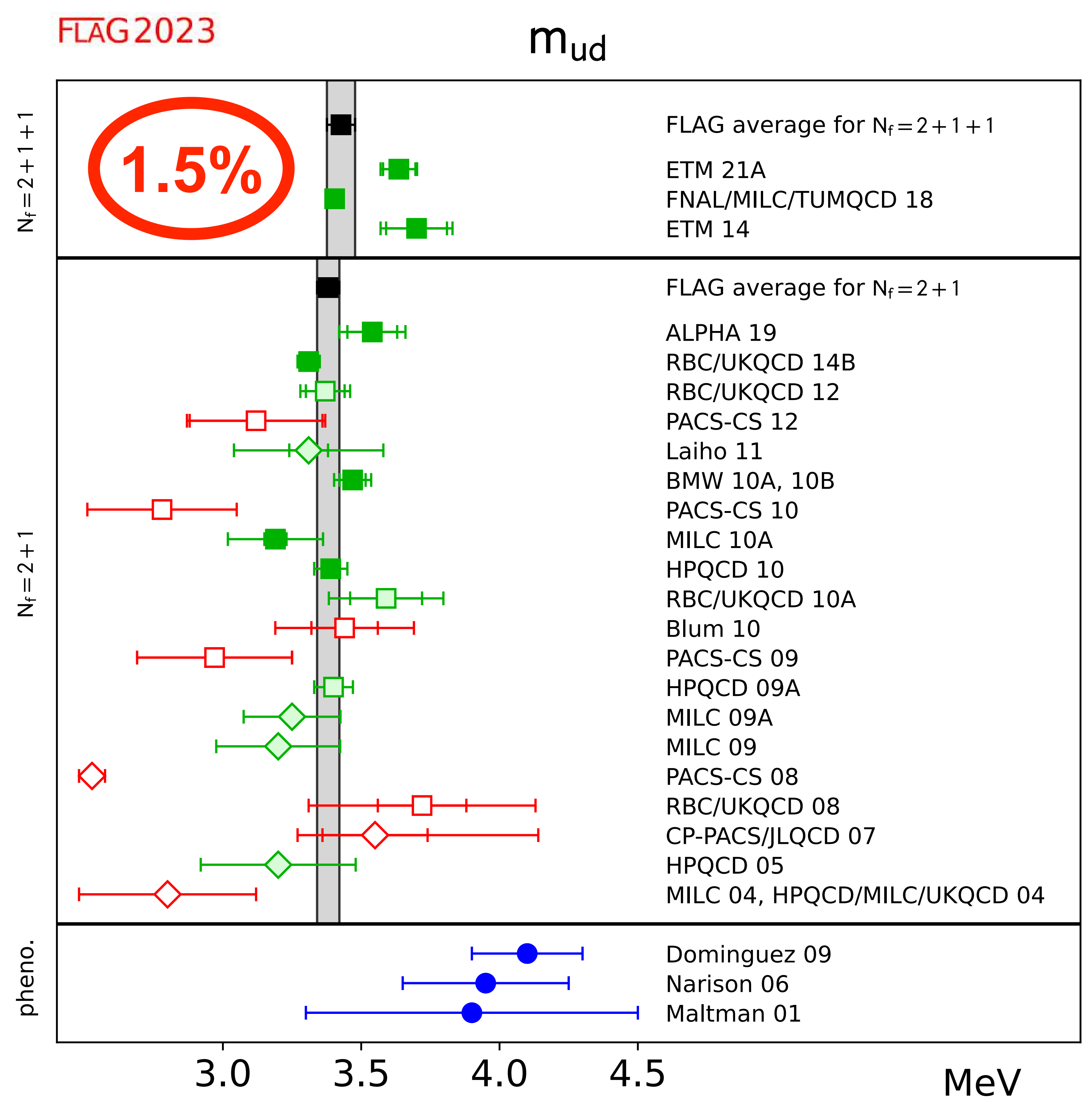


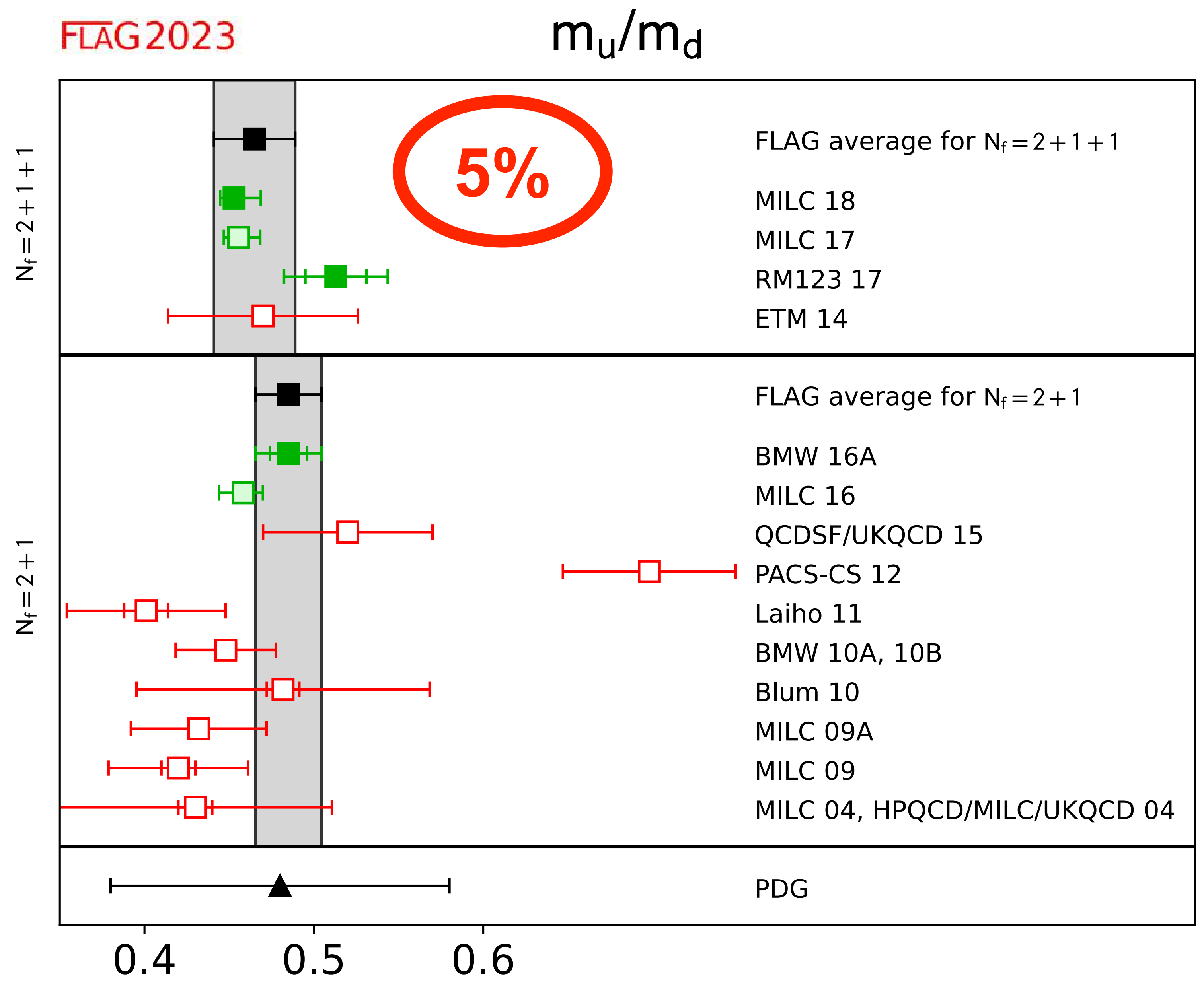


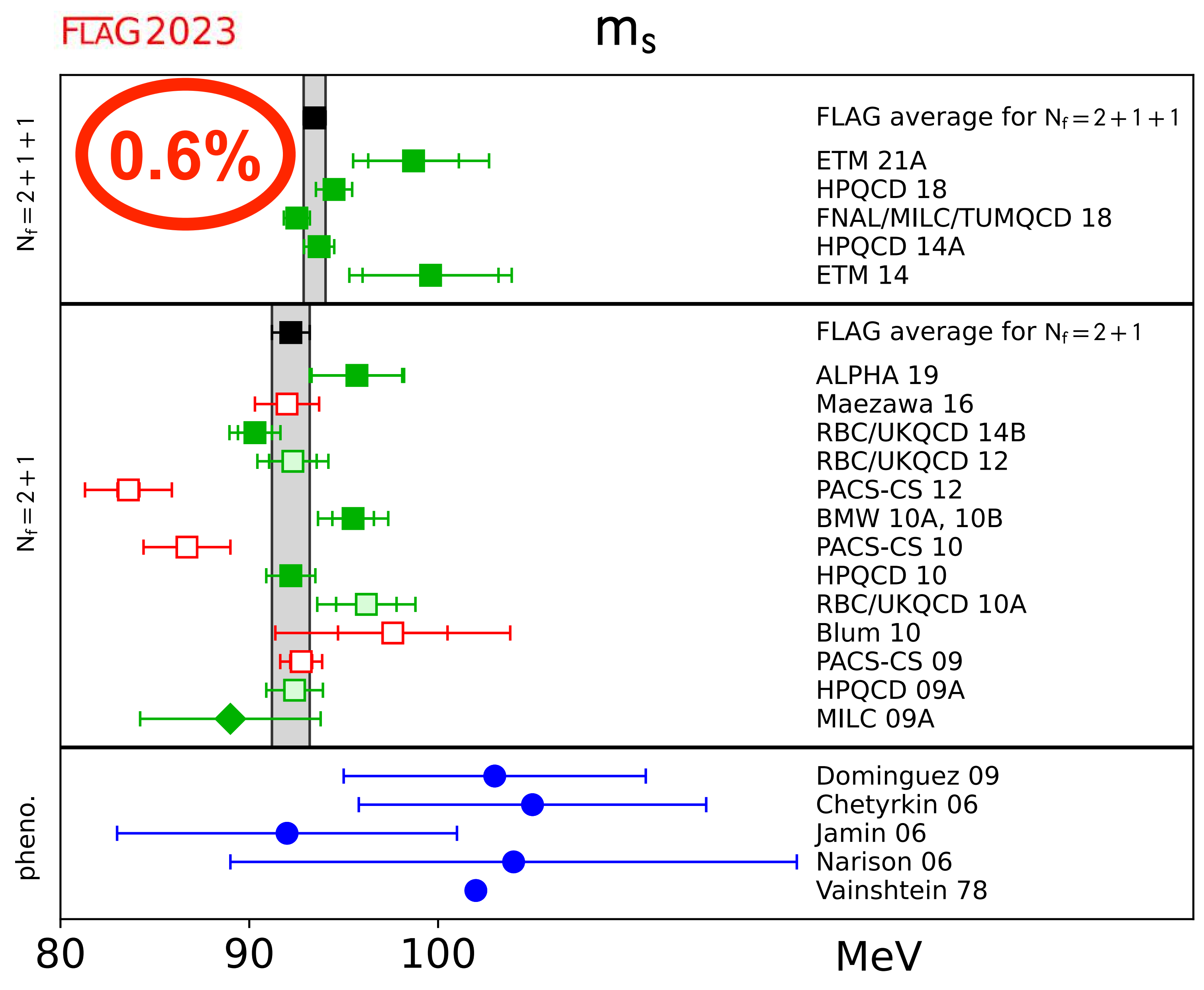
J. Huston, K. Rabbertz, G. Zanderighi
[PDG QCD review]



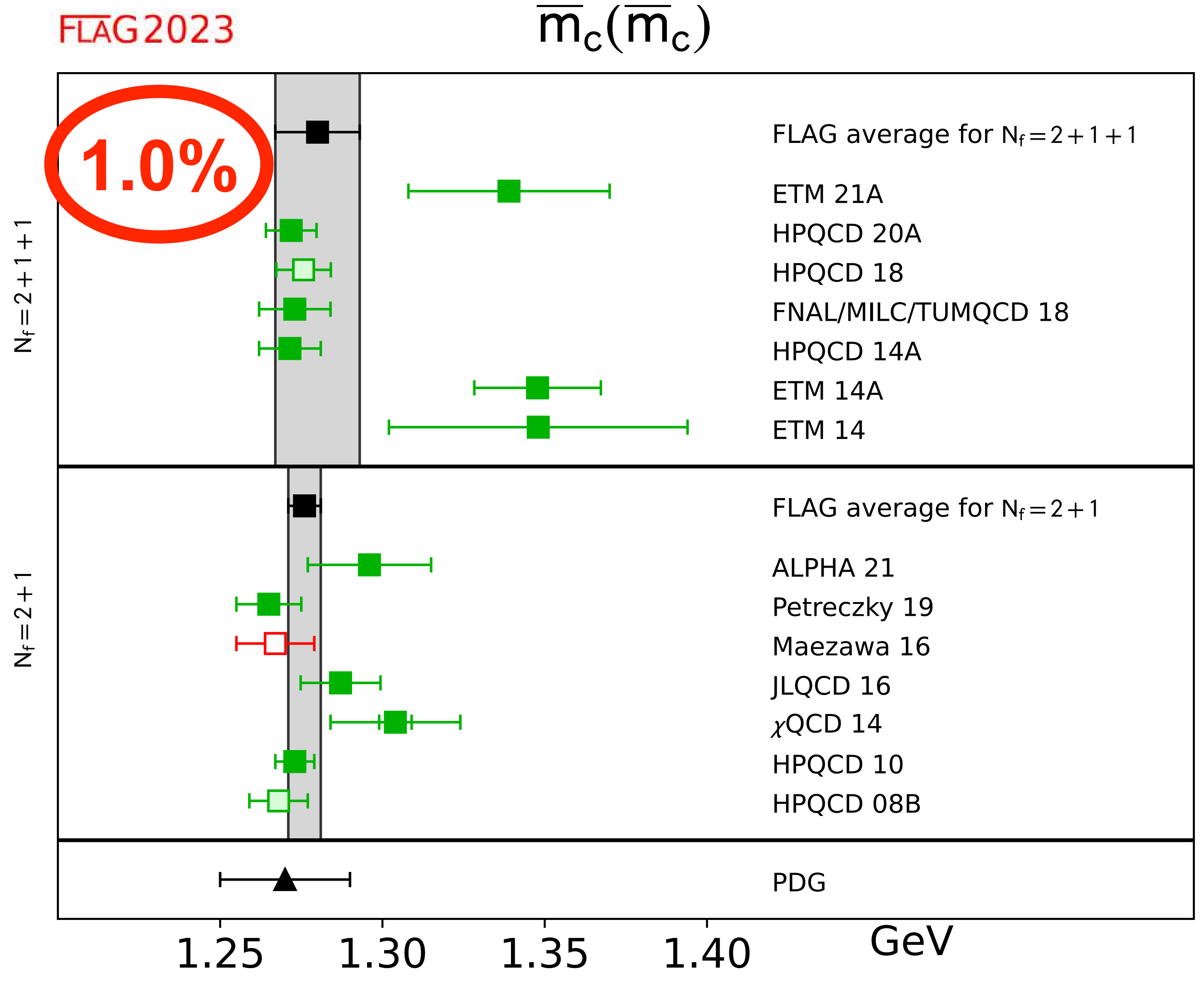
quark masses

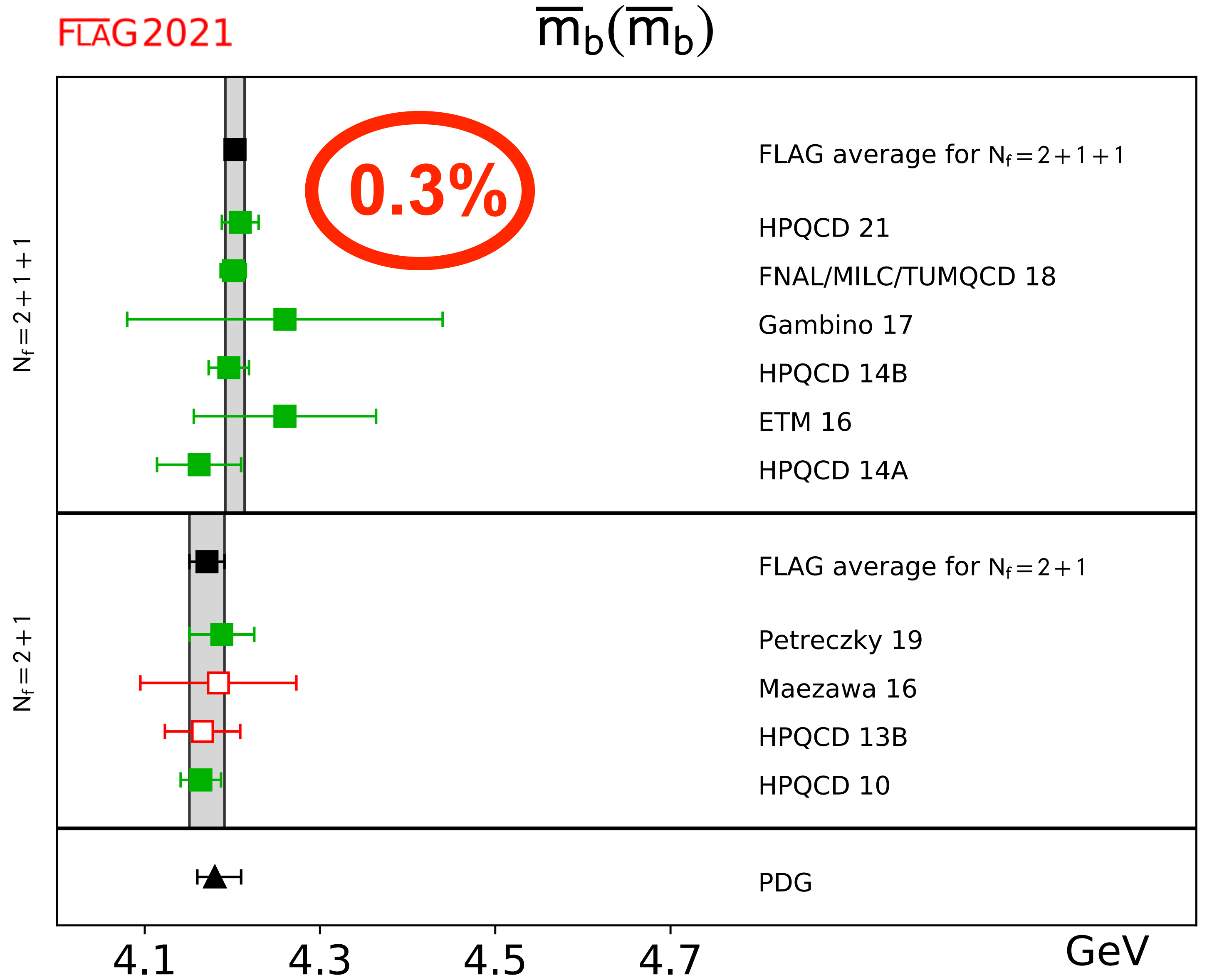




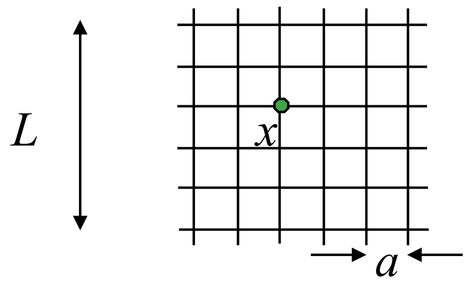


S. Aoki et al [FLAG 2021 review, arXiv:2111.09849, EPJC 2022]



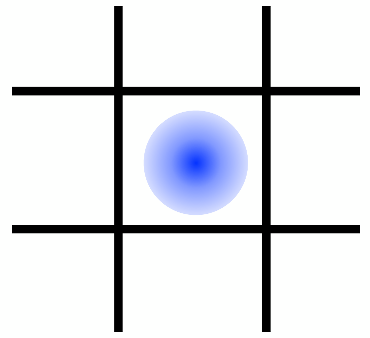


Note: PDG quark mass listings still need to be adjusted.



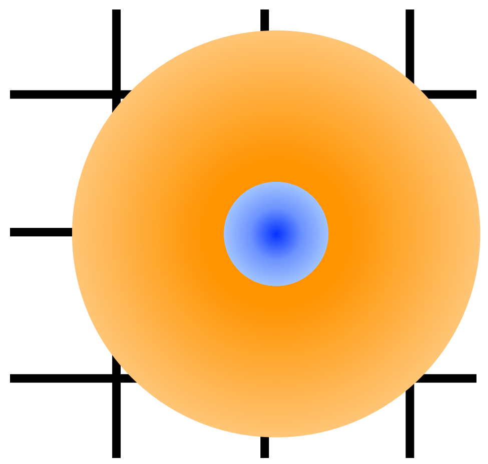
Lattice QCD Introduction: finding beauty

b quark



$m_b \gtrsim a^{-1} \gg \Lambda$ ➠ leading discretization errors $\sim (am_b)^2$
(using same action as for light quarks)

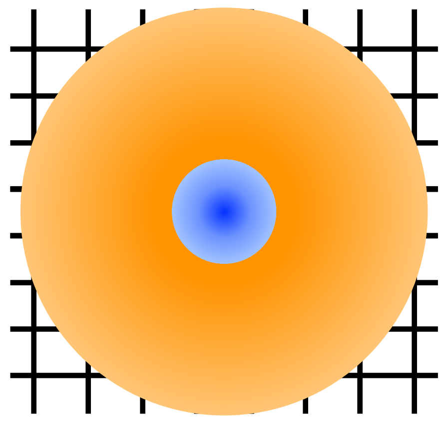
B meson



use EFT (HQET, NRQCD) ➠ Λ/m_b expansion

EFTs co-developed continuum/lattice

- lattice HQET, NRQCD: use EFT to construct lattice action
  complicated continuum limit
  nontrivial matching and renormalization ➠ (few-5)% errors
- relativistic heavy quark approach: Fermilab (1996), also Tsukuba (2003), RHQ (2006)
  matching relativistic lattice action via HQET to continuum
  nontrivial matching and renormalization ➠ (1-3)% errors

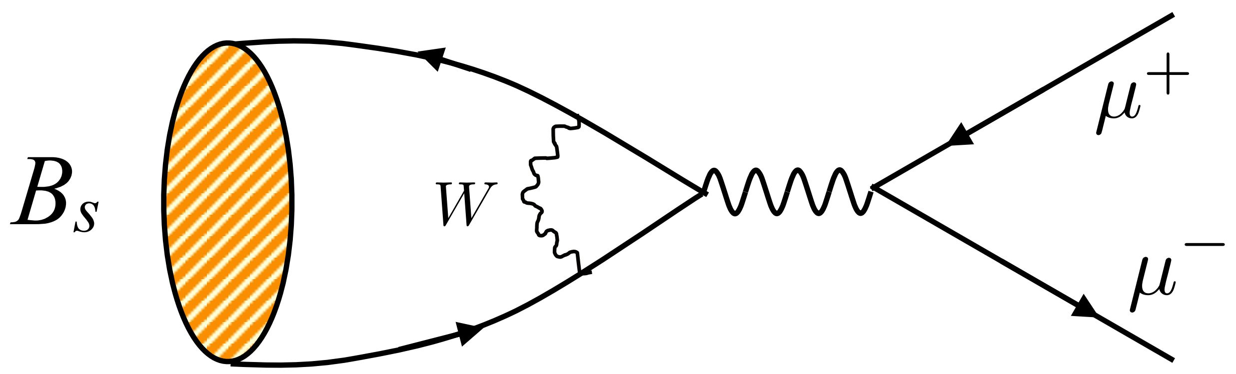


$a^{-1} > m_b \gg \Lambda$ + highly improved light quark action
➠ same action for all quarks
➠ simple renormalization (Ward identities) ➠ < 1% errors

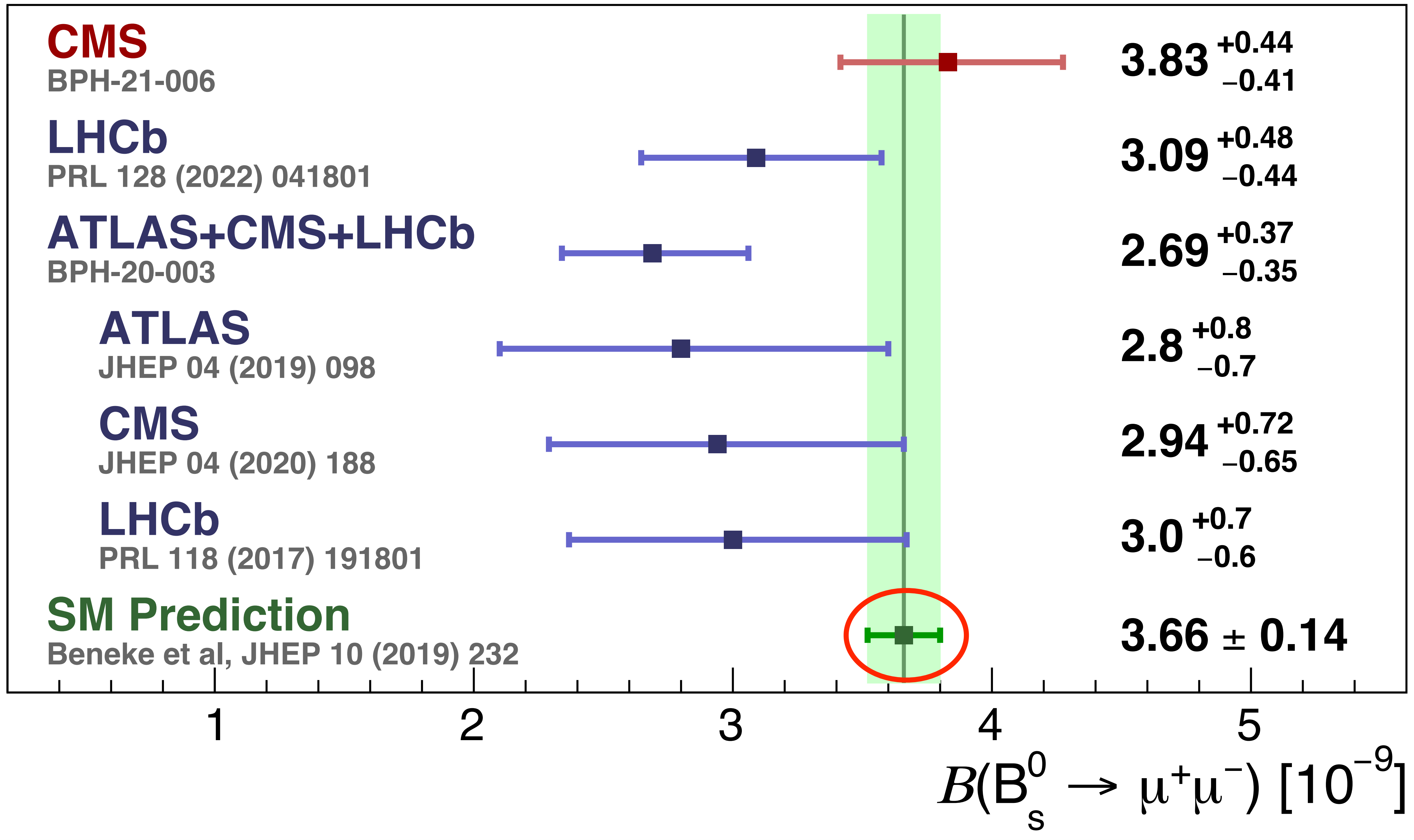
Rare leptonic decay $B_s \to \mu\mu$

SM prediction for rare leptonic decay rate

[Beneke et al, arXiv:1908.07011, JHEP 2019]



Silvano Tosi @ LHCP2023



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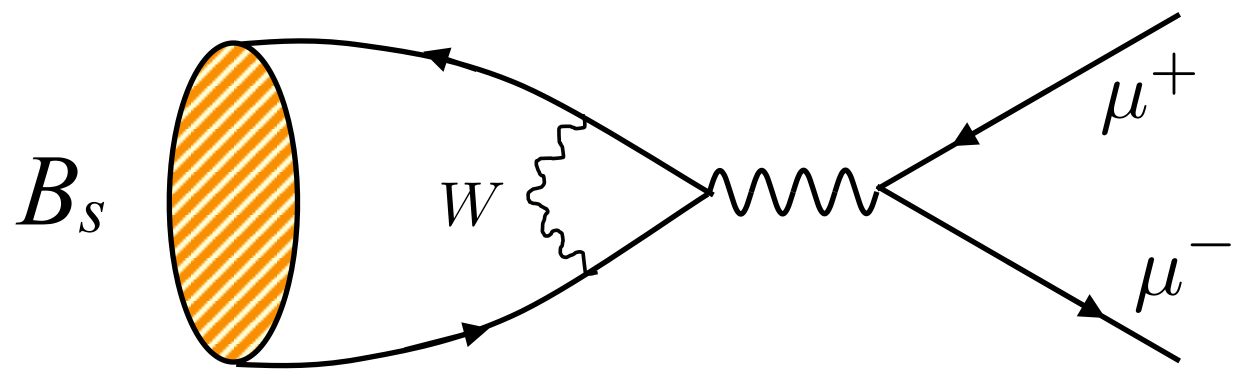
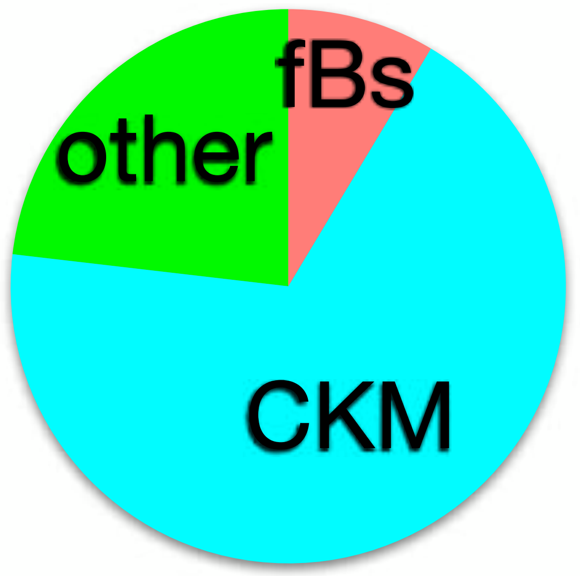


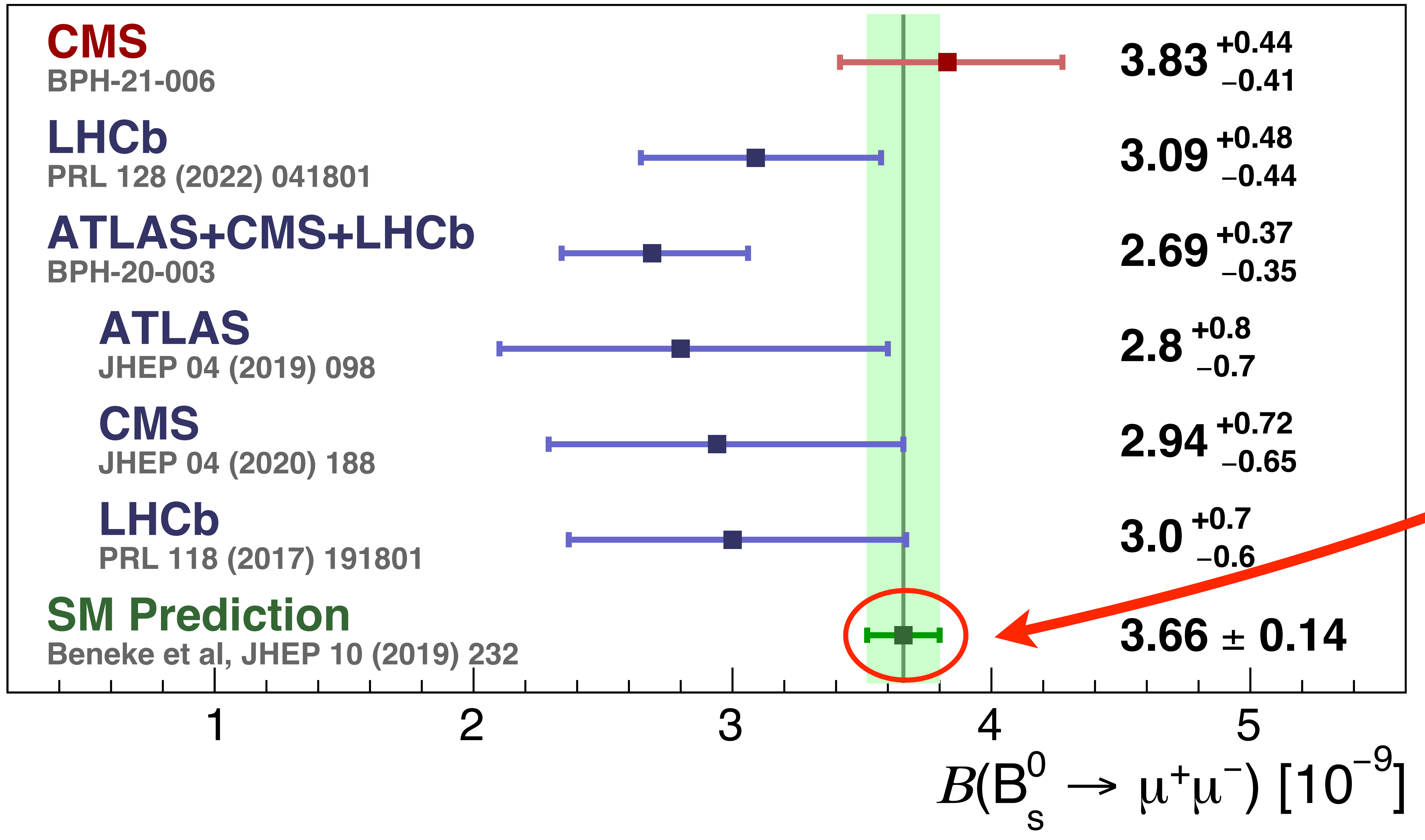
error²

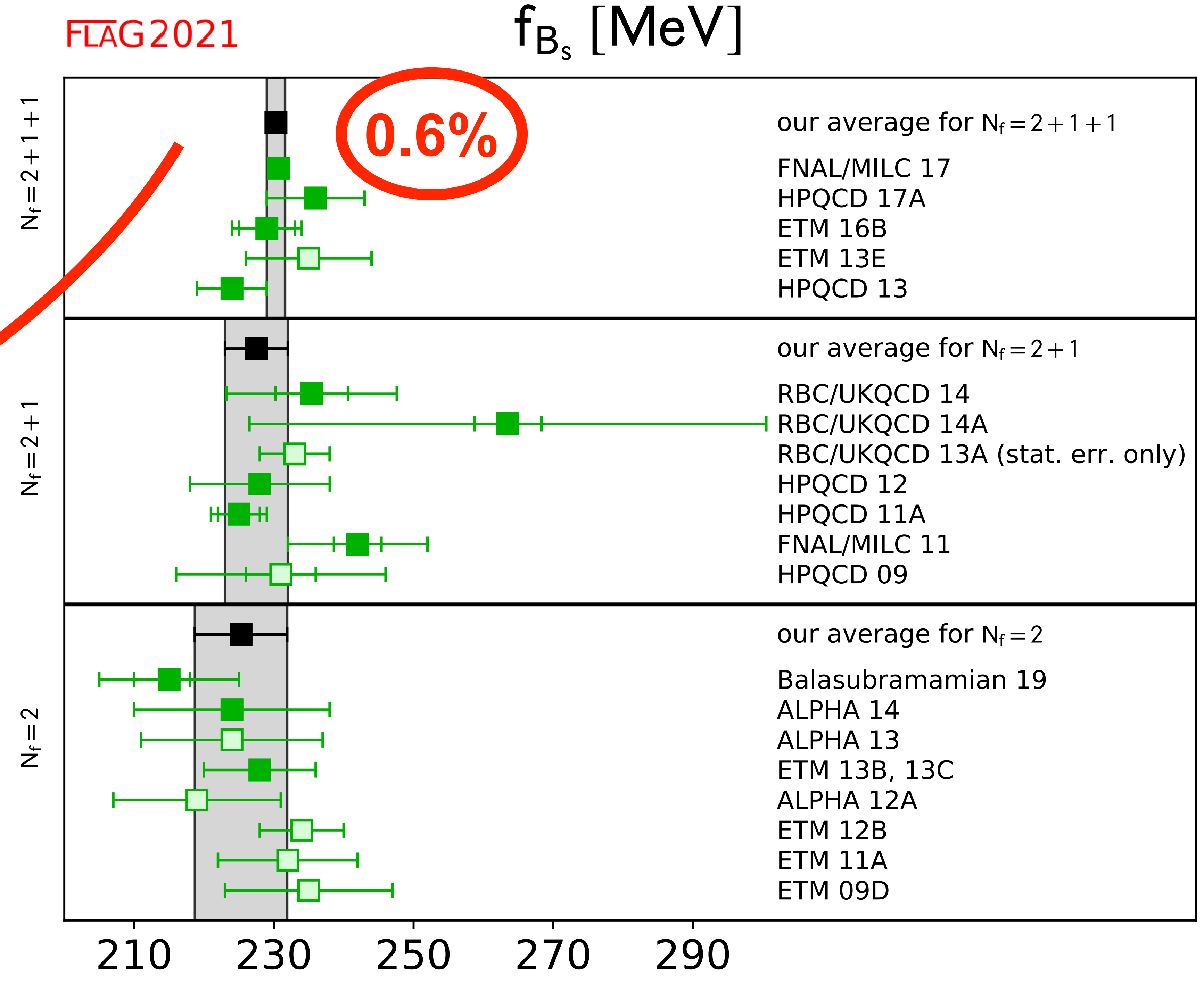


- includes structure-dependent QED corrections
- dominant uncertainty due to $|V_{cb}|$
- LQCD decay constant sub dominant source of uncertainty

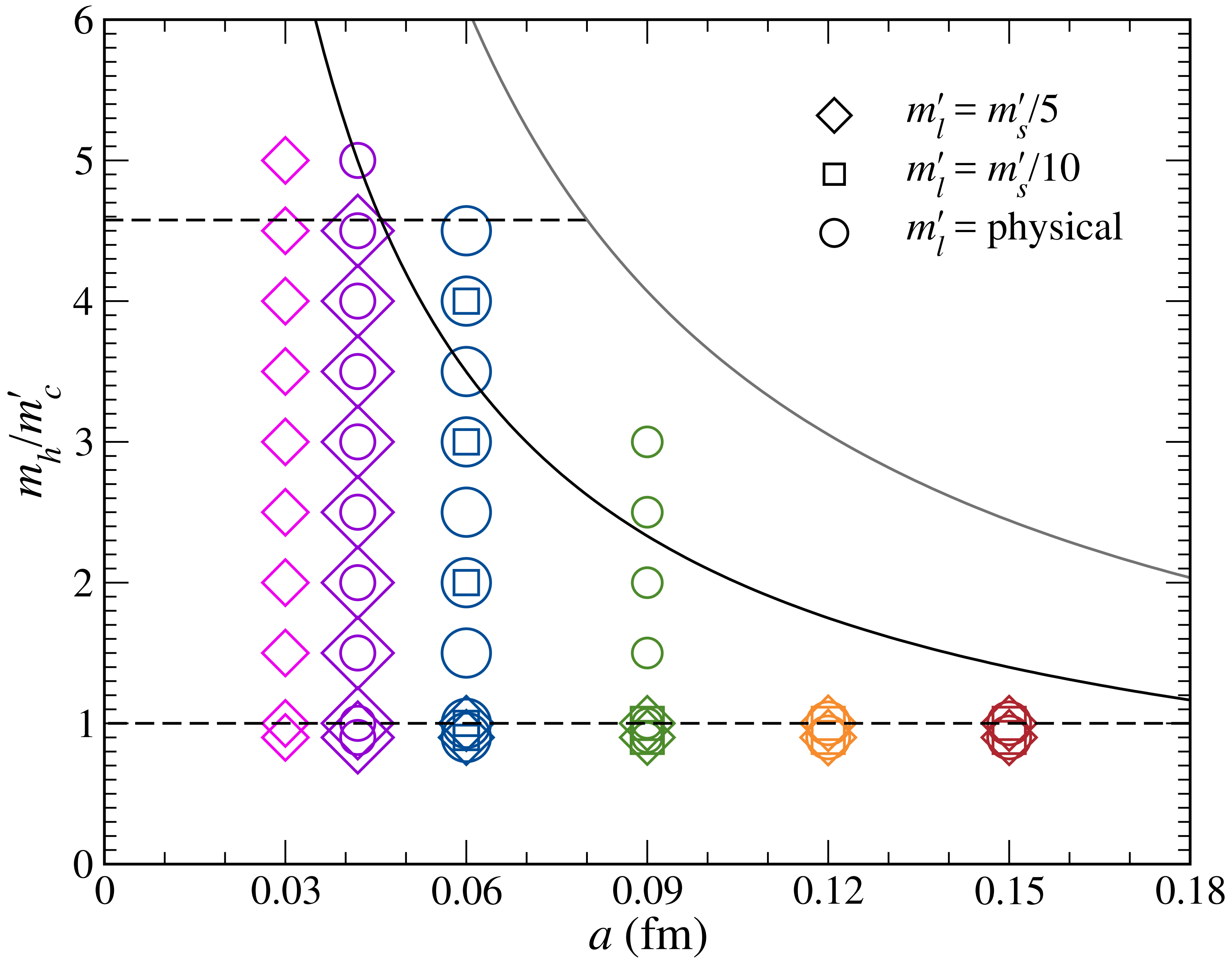
S. Aoki et al [FLAG 2021 review, arXiv:2111.09849, EPJC 2022]

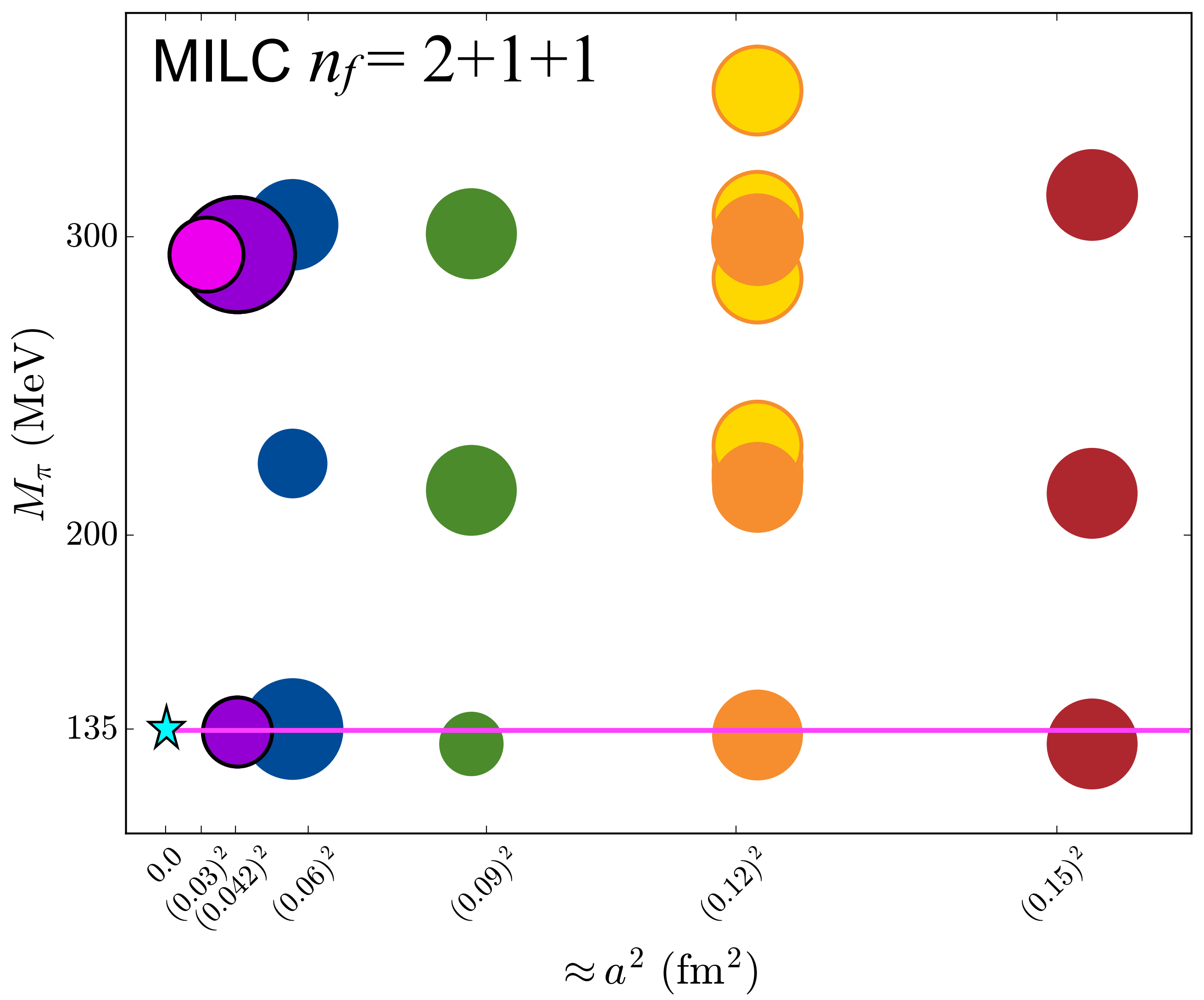
Silvano Tosi @ LHCP2023



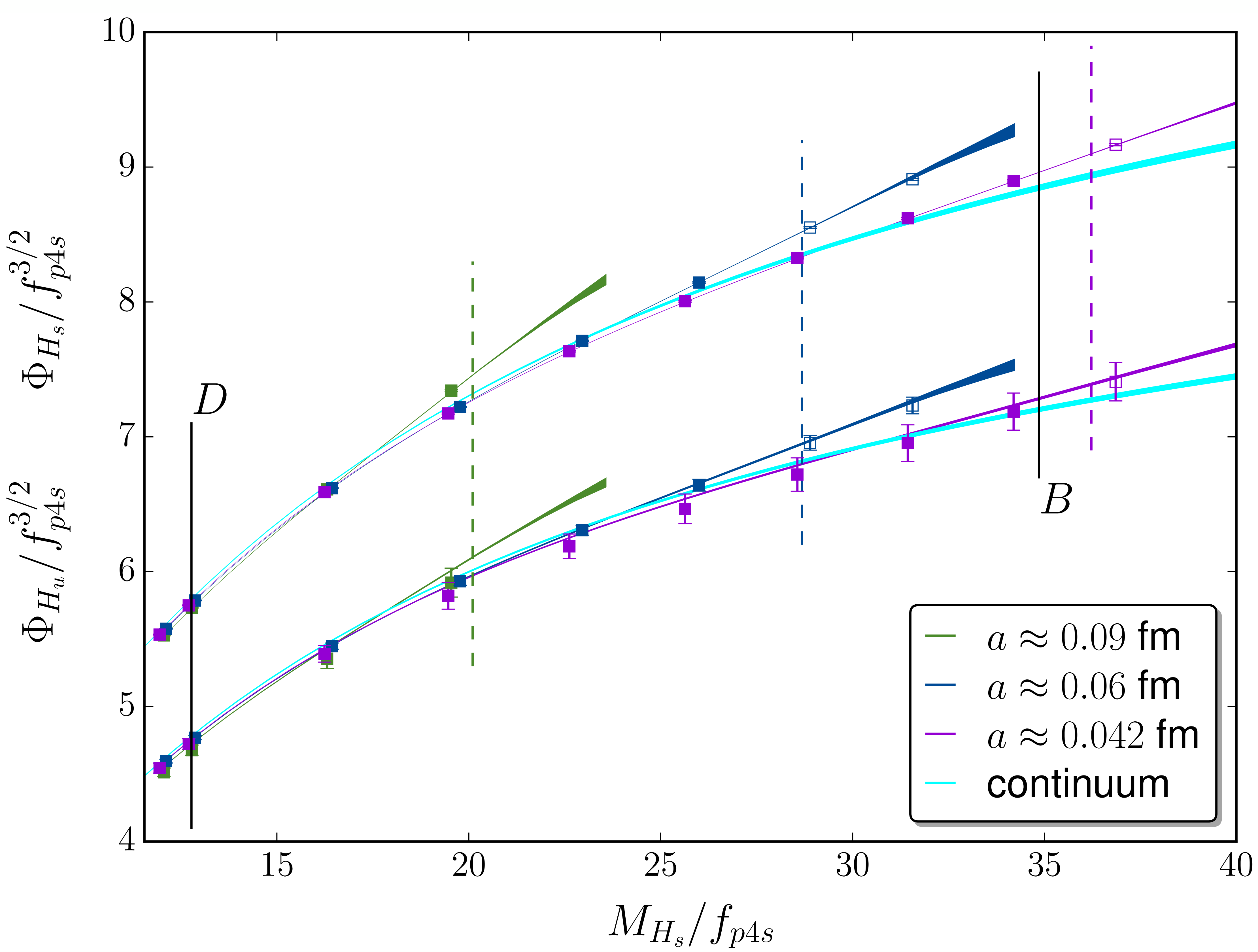


B, D meson decay constant results





A. Bazavov et al [FNAL/MILC, arXiv:1712.09262, 2018 PRD]

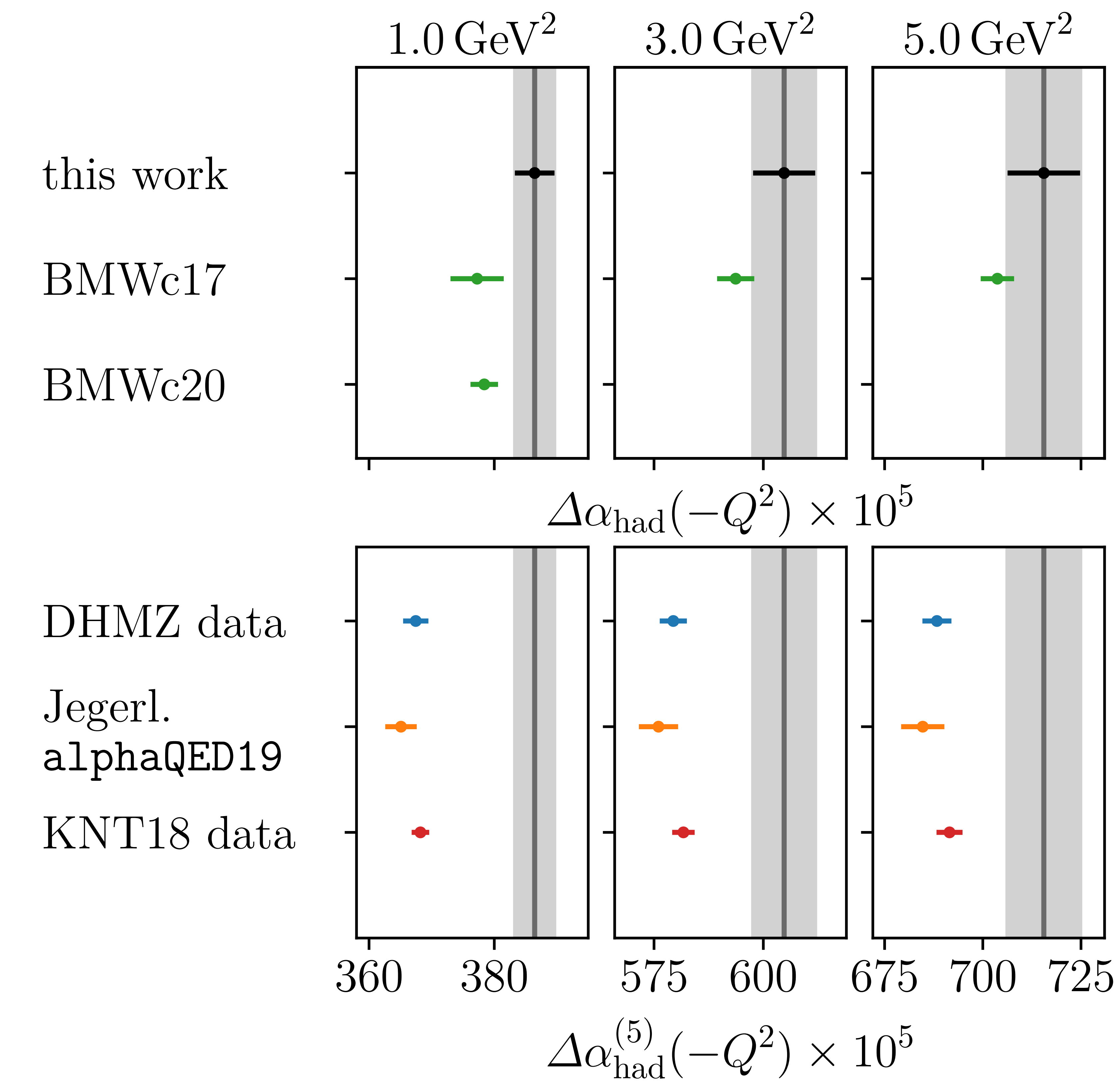


small errors due to **physical light quark masses**
improved quark action with small discretization errors even for heavy quarks
no renormalization (Ward identity)

Connections

$$\sigma(e^+e^- \to \text{hadrons}) \Leftrightarrow a_\mu^{\text{HVP}} \Leftrightarrow \Delta\alpha_{\text{had}}(M_Z^2)$$

H. Wittig @ Higgscentre workshop



Dispersion integral: $\Delta\alpha^{(5)}_{\text{had}}(q^2) = -\frac{\alpha\, q^2}{3\pi} \oint_{m^2_{\pi^0}}^{\infty} ds\, \frac{R(s)}{s(s-q^2)}$

Lattice QCD:

$$\Delta\alpha_{\text{had}}(-Q^2) = \frac{\alpha}{\pi}\frac{1}{Q^2}\int_0^\infty dt\, G(t)\left[Q^2t^2 - 4\sin^2\left(\tfrac{1}{2}Q^2t^2\right)\right]$$

- Direct lattice calculation of $\Delta\alpha(-Q^2)$ on the same gauge ensembles used in Mainz/CLS 22
  [Cè et al., JHEP 08 (2022) 220, arXiv:2203.08676]
- Tension of $\sim 3\sigma$ observed with data-driven evaluation of $\Delta\alpha_{\text{had}}(-Q^2)$ for $Q^2 \gtrsim 3\,\text{GeV}^2$
  → consistent with tension for window observable

Connections

$$\sigma(e^+e^- \to \text{hadrons}) \quad \Leftrightarrow \quad a_\mu^{\text{HVP}} \quad \Leftrightarrow \quad \Delta\alpha_{\text{had}}(M_Z^2)$$

H. Wittig @ Higgscentre workshop

Adler function approach, aka. “Euclidean split technique”

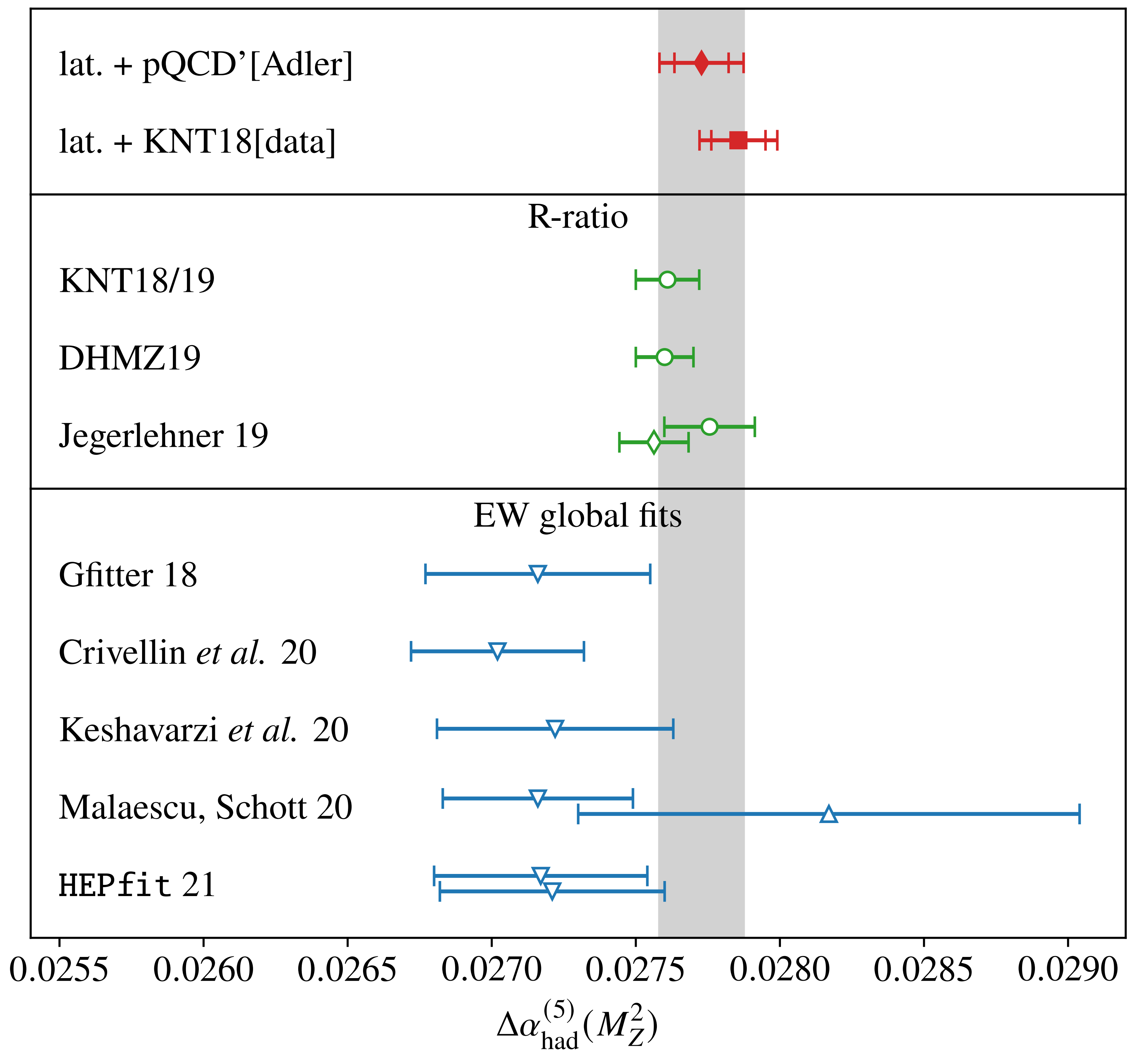
$$\Delta\alpha^{(5)}_{\text{had}}(M_Z^2) = \Delta\alpha^{(5)}_{\text{had}}(-Q_0^2)$$

$$+[\Delta\alpha^{(5)}_{\text{had}}(-M_Z^2) - \Delta\alpha^{(5)}_{\text{had}}(-Q_0^2)]$$

$$+[\Delta\alpha^{(5)}_{\text{had}}(M_Z^2) - \Delta\alpha^{(5)}_{\text{had}}(-M_Z^2)]$$

$$\Rightarrow \quad \Delta\alpha^{(5)}_{\text{had}}(M_Z^2) = 0.027\,73(9)_{\text{lat}}(2)_{\text{btm}}(12)_{\text{pQCD}}$$

[Cè et al., JHEP 08 (2022) 220, arXiv:2203.08676]



- Agreement between lattice QCD and evaluations based on the R-ratio

Connections

$$\sigma(e^+e^- \to \text{hadrons}) \quad \Leftrightarrow \quad a_\mu^{\text{HVP}} \quad \Leftrightarrow \quad \Delta\alpha_{\text{had}}(M_Z^2)$$

- $\Delta\alpha_{\text{had}}(M_Z^2)$ also depends on the hadronic vacuum polarization function, and can be written as an integral over $\sigma(e^+e^- \to \text{hadrons})$, but weighted towards higher energies.
- a shift in a_μ^{HVP} also changes $\Delta\alpha_{\text{had}}(M_Z^2)$: ➠ EW fits [Passera, et al, 2008, Crivellin et al 2020, Keshavarsi et al 2020, Malaescu & Scott 2020]
  If the shift in a_μ^{HVP} is in the low-energy region ($\lesssim 1\,\text{GeV}$), the impact on $\Delta\alpha_{\text{had}}(M_Z^2)$ and EW fits is small.
- A shift in a_μ^{HVP} from low ($\lesssim 2\,\text{GeV}$) energies
  ➠ $\sigma(e^+e^- \to \pi\pi)$
  must satisfy unitarity & analyticity constraints ➠ $F_\pi^V(s)$
  can be tested with lattice calculations
  [Colangelo, Hoferichter, Stoffer, arXiv:2010.07943]

Peter Stoffer @ Lattice HVP workshop

Constraints on the two-pion contribution to HVP arXiv:2010.07943 [hep-ph]

Modifying $a_\mu^{\pi\pi}|_{\leq 1\,\text{GeV}}$

- "low-energy" scenario: local changes in cross section of $\sim 8\%$ **around ρ**
- "high-energy" scenario: impact on **pion charge radius** and space-like VFF $\Rightarrow$ chance for **independent lattice-QCD checks**
- requires **factor ~ 3 improvement** over χQCD result:
  $\langle r_\pi^2 \rangle = 0.433(9)(13)\,\text{fm}^2$
  → arXiv:2006.05431 [hep-ph]