Lattice QCD calculation of the hadronic light-by-light contribution to the magnetic moment of the muon

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Cluster of Excellence

$(g-2)_\mu$: a history of testing the Standard Model

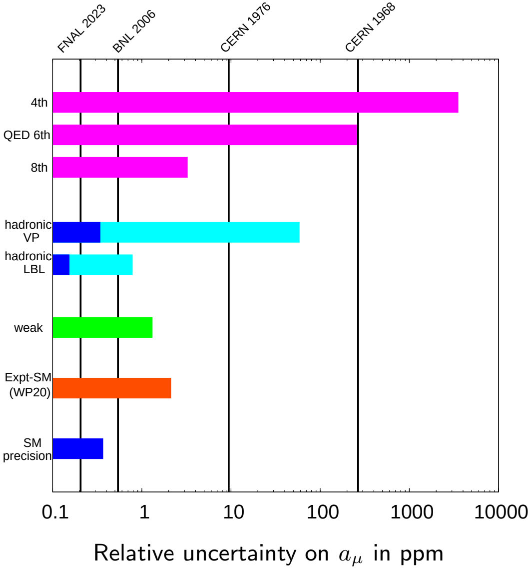
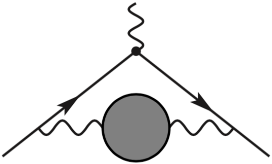
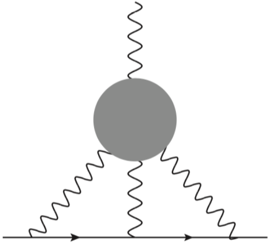


Figure inspired by [Jegerlehner 1705.00263].

Source of dominant uncertainties in SM



Hadronic vacuum polarisation



Hadronic light-by-light scattering

HVP: O(α^2), about $700 \cdot 10^{-10}$

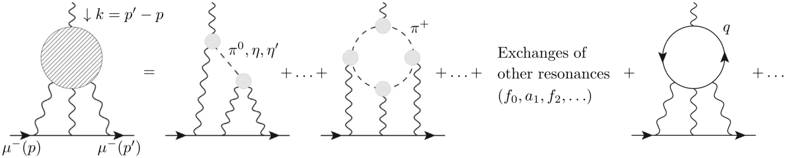
WP20 precision: 0.6%

Desirable precision: 0.2%

HLbL: O(α^3), about $10 \cdot 10^{-10}$

WP20 precision: 20%.

Desirable precision: 10%.



Wisdom gained from model calculations Prades, de Rafael, Vainshtein 0901.0306

- heavy (charm) quark loop makes a small contribution

$$a_\mu^{\rm HLbL} = (\frac{\alpha}{\pi})^3\, N_c \mathcal{Q}_c^4 c_4 \frac{m_\mu^2}{m_c^2} + \dots, \qquad c_4 \approx 0.62.$$

- Light-quarks: (A) charged pion loop is negative, proportional to m_π^{-2}:

$$a_\mu^{\rm HLbL} = (\frac{\alpha}{\pi})^3\, c_2 \frac{m_\mu^2}{m_\pi^2} + \dots, \qquad c_2 \approx -0.065.$$

  (B) The neutral-pion exchange is positive, $\log^2(m_\pi^{-1})$ divergent:
  Knecht, Nyffeler, Perrottet, de Rafael PRL88 (2002) 071802

$$a_\mu^{\rm HLbL} = (\frac{\alpha}{\pi})^3\, N_c \frac{m_\mu^2}{48\pi^2(F_\pi^2/N_c)} \Big[\log^2 \frac{m_\rho}{m_\pi} + {\rm O}\Big(\log \frac{m_\rho}{m_\pi}\Big) + {\rm O}(1)\Big].$$

- For real-world quark masses: using form factors for the mesons is essential, and resonances up to 1.5 GeV are still relevant ⇒ medium-energy QCD.

Approaches to a_μ^{HLbL}

1. **Model calculations:** (the only approach until 2014)
   - based on pole- and loop-contributions of hadron resonances

2. **Dispersive representation:** the Bern approach has been worked out furthest.
   - identify and compute contributions of most important intermediate states
   - determine/constrain the required input (transition form factors, $\gamma^*\gamma^* \to \pi\pi$ amplitudes, ...) dispersively

3. **Experimental program:** provide input for dispersive approach, e.g. $(\pi^0, \eta, \eta') \to \gamma^{(*)}\gamma^{(*)}$ at virtualities $Q^2 \lesssim 3\,\mathrm{GeV}^2$; active program (see talk by Andrzej Kupsc)

4. **Lattice calculations:**
   a) Direct, inclusive calculation:
   - RBC-UKQCD T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, Ch. Lehner, ...
   - Mainz N. Asmussen, E.-H. Chao, A. Gérardin, J. Green, J. Hudspith, HM, A. Nyffeler
   - Recent work by BMW and ETM collaborations.

   b) Provide input for dispersive approach: π^0, η, η' transition form factors, $\langle AVV \rangle$ correlator, ...

Plan

Direct calculation:

- why it's even possible in Euclidean space;
- why use a coordinate-space representation;
- computing the QED side of the diagram;
- some insight in the challenges of the lattice calculation;
- current status.

Calculations of the transition form factors of pseudoscalar mesons:

- methodology;
- neutral pion;
- η, η'.

Connection between Minkowskian and Euclidean correlation functions

Start from the vacuum expectation value of a (Minkowski-)time-ordered product of scalar fields,

$$\langle 0 \big| \mathrm{T}\big\{\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\big\} \big| 0 \rangle.$$

In Fourier representation:

$$\langle 0 \big| \mathrm{T}\big(\prod_{a=1}^{4} \phi_a(x_a) \big) \big| 0 \rangle = \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \frac{d^4p_3}{(2\pi)^4}\, \Pi(p_1,p_2,p_3)\, \exp\Big(-i \sum_{a=1}^{3} p_a \cdot \Delta x_a \Big),$$

where $\Delta x_a = x_a - x_4$.

NB. I use the $(+ - - -)$ Minkowski metric.

Consider the function

$$f(z) = \langle 0 \big| \mathrm{T}\big(\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\big) \big| 0 \rangle \Big|_{x_a^0 \to z x_a^0}$$

where all time components of the coordinates are simultaneously multiplied by a real variable z . In the Fourier representation,

$$\begin{aligned} f(z) &= \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\, \Pi(p_1,p_2,p_3)\, \exp\Big(i\sum_{a=1,2,3}(-z p_a^0 \Delta x_a^0 + \boldsymbol{p}_a\cdot\Delta\boldsymbol{x}_a)\Big) \\ &= \frac{1}{z^3}\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\Pi\Big((-\frac{p_1^0}{z},\boldsymbol{p}_1),(-\frac{p_2^0}{z},\boldsymbol{p}_2),(-\frac{p_3^0}{z},\boldsymbol{p}_3)\Big) \\ &\qquad \times \exp\Big(i\sum_{a=1,2,3}(p_a^0\Delta x_a^0 + \boldsymbol{p}_a\cdot\Delta\boldsymbol{x}_a)\Big). \end{aligned}$$

In the second equality we have made the change of integration variables $p_a^0 \to -\frac{1}{z}p_a^0$. Now extend the function to $\mathrm{Im}(z) \le 0$ and even set $z = -i$,

$$\begin{aligned} f(-i) &= -i\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\Pi\Big((-ip_1^0,\boldsymbol{p}_1),(-ip_2^0,\boldsymbol{p}_2),(-ip_3^0,\boldsymbol{p}_3)\Big) \\ &\qquad \times \exp\Big(i\sum_{a=1,2,3}(p_a^0\Delta x_a^0 + \boldsymbol{p}_a\cdot\Delta\boldsymbol{x}_a)\Big). \end{aligned}$$

The Euclidean correlator is precisely this quantity,

$$\Big\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\Big\rangle_{\text{Eucl.path integral}} = f(-i).$$

Kinematic reach of Euclidean correlation functions

$$-i\Pi\Big((-ip_1^0,\boldsymbol{p}_1),(-ip_2^0,\boldsymbol{p}_2),(-ip_3^0,\boldsymbol{p}_3);\Big) = \int d^4x_1 d^4x_2 d^4x_3$$
$$\exp\Big(-i\sum_{a=1,2,3}(p_a^0\Delta x_a^0+\boldsymbol{p}_a\cdot\Delta\boldsymbol{x}_a)\Big)\Big\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\Big\rangle_{\rm E}.$$

Lorentz invariance: actually, $\Pi(p_1,p_2,p_3)$ depends only on 6 invariants, $(p_a\cdot p_b)_{\rm Minkowski}$.

The Euclidean correlator, Fourier-transformed as in the Eq. above with momenta p_a, gives you the function Π with arguments

$$(-ip_{a,0})(-ip_{b,0})+\boldsymbol{p}_a\cdot\boldsymbol{p}_b = -(p_a\cdot p_b)_{\rm Minkowski}\in\mathbb{R}.$$

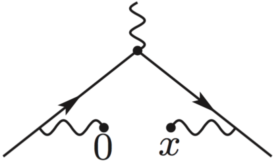
For instance, if in Euclidean you choose $p_1=Q$, corresponding to an incoming 'photon', you obtain the amplitude Π for a photon virtuality $-Q^2$, i.e. spacelike.

More generally, from Euclidean you only get the 'photon' four-point amplitude at kinematics such that '$\gamma\gamma$'$\rightarrow$ hadrons is not possible.

Selected literature

1. Hayakawa, Blum, Izubuchi, Yamada hep-lat/0509016 (LAT'05, Dublin); 1407.2923.
2. Blum et al. 1510.0710; 1610.0460; 1911.0812 (results with QED in finite volume);
3. Blum et al. 1705.0106 (QED in infinite volume, tested on free quark loop computed on the lattice); Blum et al. 2304.04423 (results);
4. Mainz group conference proceedings: 1510.08384, 1609.08454, 1711.02466, 1801.04238, 1811.08320, 1911.05573.
5. Mainz group: 2006.16224 (at $\mathrm{SU(3)_f}$ symmetric point); 2104.02632 (extrapolating to physical quark masses); 2204.08844 (charm contribution).
6. Mainz QED kernel: 2210.12263. Available at `https://github.com/RJHudspith/KQED`
7. 2311.10628 Zimmermann, Gérardin;

Analogy: hadronic vacuum polarization in x-space HM 1706.01139



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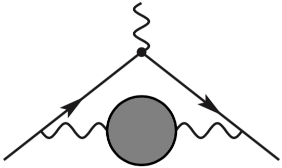


QED kernel $H_{\mu\nu}(x)$ $\qquad$ $a_\mu^{\rm hvp}$

$$a_\mu^{\rm hvp} = \int d^4x\; H_{\mu\nu}(x)\, \Big\langle j_\mu(x) j_\nu(0)\Big\rangle_{\rm QCD},$$

$$j_\mu = \frac{2}{3}\bar u\gamma_\mu u - \frac{1}{3}\bar d\gamma_\mu d - \frac{1}{3}\bar s\gamma_\mu s + \dots; \qquad H_{\mu\nu}(x) = -\delta_{\mu\nu}\mathcal{H}_1(|x|) + \frac{x_\mu x_\nu}{x^2}\,\mathcal{H}_2(|x|)$$

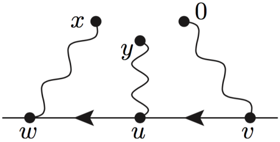
Kernel known in terms of Meijer's functions: $\mathcal{H}_i(|x|) = \frac{8\alpha^2}{3m_\mu^2} f_i(m_\mu|x|)$ with

$$f_2(z) = \frac{G^{2,2}_{2,4}\left(z^2\Big|\,\begin{matrix}\frac{7}{2},4\\ 4,5,1,1\end{matrix}\right) - G^{2,2}_{2,4}\left(z^2\Big|\,\begin{matrix}\frac{7}{2},4\\ 4,5,0,2\end{matrix}\right)}{8\sqrt{\pi}z^4},$$

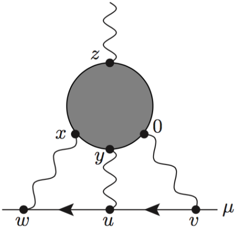
$$f_1(z) = f_2(z) - \frac{3}{16\sqrt{\pi}}\cdot\left[G^{2,3}_{3,5}\left(z^2\Big|\,\begin{matrix}1,\frac{3}{2},2\\ 2,3,-2,0,0\end{matrix}\right) - G^{2,3}_{3,5}\left(z^2\Big|\,\begin{matrix}1,\frac{3}{2},2\\ 2,3,-1,-1,0\end{matrix}\right)\right].$$

Coordinate-space approach to $a_\mu^{\rm HLbL}$, Mainz version

QED kernel $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$



$\Rightarrow$



$$a_\mu^{\rm HLbL} = \frac{me^6}{3} \underbrace{\int d^4y}_{=2\pi^2|y|^3 d|y|} \Big[\int d^4x \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\rm QED}\; \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\rm =QCD\ blob}\Big].$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z\, z_\rho \Big\langle\, j_\mu(x)\, j_\nu(y)\, j_\sigma(z)\, j_\lambda(0)\Big\rangle.$$

- $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ computed in the continuum & infinite-volume
- no power-law finite-volume effects & only a 1d integral to sample the integrand in $|y|$.

[Asmussen, Gérardin, Green, HM, Nyffeler 1510.08384, 1609.08454, 2210.12263 (JHEP).]

Sketch of derivation: starting point in Euclidean space

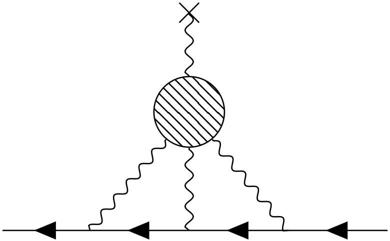
With $j_\rho(x) = \frac{2}{3}(\bar{u}\gamma_\rho u)(x) - \frac{1}{3}(\bar{d}\gamma_\rho d)(x) - \frac{1}{3}(\bar{s}\gamma_\rho s)(x)$,
we consider the matrix element

$$(ie)\langle\mu^-(p')|j_\rho(0)|\mu^-(p)\rangle = -(ie)\bar{u}(p')\left[\gamma_\rho F_1(k^2) + \frac{\sigma_{\rho\sigma}k_\sigma}{2m}F_2(k^2)\right]u(p).$$

- e is the electric charge of the electron
- m is the muon mass,
- $\sigma_{\rho\sigma} \equiv \frac{i}{2}[\gamma_\rho, \gamma_\sigma]$
- $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, $\gamma_\mu^\dagger = \gamma_\mu$.
- $k_\mu = p'_\mu - p_\mu$

The anomalous magnetic moment is then given by the Pauli form factor at vanishing momentum transfer $\boxed{a_\mu = F_2(0)}$.

The HLbL diagram



The diagram corresponds to the Euclidean momentum-space integrals

$$(ie)\langle\mu^-(p')|j_\rho(0)|\mu^-(p)\rangle = (-ie)^3\,(ie)^4\int_{q_1,q_2}\frac{1}{q_1^2q_2^2(q_1+q_2-k)^2}$$
$$\times\frac{-1}{(p'-q_1)^2+m^2}\;\frac{-1}{(p'-q_1-q_2)^2+m^2}$$
$$\times\,\bar u(p')\gamma_\mu(i\not{p}'-i\not{q}_1-m)\gamma_\nu(i\not{p}'-i\not{q}_1-i\not{q}_2-m)\gamma_\lambda u(p)$$
$$\times\,\Pi_{\mu\nu\lambda\rho}(q_1,q_2,k-q_1-q_2),$$

with the QCD four-point correlation function ($\int_x \equiv \int d^4x$)

$$\Pi_{\mu\nu\lambda\rho}(q_1,q_2,q_3)=\int_{x,y,z}e^{-i(q_1\cdot x+q_2\cdot y+q_3\cdot z)}\Big\langle j_\mu(x)j_\nu(y)j_\lambda(z)j_\rho(0)\Big\rangle_{\rm QCD}.$$

The projection formula

Current conservation implies [Aldins et al Phys. Rev. D 1 (1970) 2378]

$$\Pi_{\mu\nu\lambda\rho}(q_1, q_2, k - q_1 - q_2) = -k_\sigma \frac{\partial}{\partial k_\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2),$$

which can be used to show

$$a_\mu^{\rm HLbL} = F_2(0) = \frac{-i}{48m} \mathrm{Tr}\{[\gamma_\rho, \gamma_\sigma](-i\not{p} + m)\Gamma_{\rho\sigma}(p,p)(-i\not{p} + m)\}\Big|_{p^2=-m^2} .$$

The HLbL contribution to the vertex function reads

$$\begin{aligned}\Gamma_{\rho\sigma}(p', p) \quad &= \quad -e^6 \int_{q_1,q_2} \frac{1}{q_1^2 q_2^2 (q_1 + q_2 - k)^2} \frac{1}{(p' - q_1)^2 + m^2} \frac{1}{(p' - q_1 - q_2)^2 + m^2} \\ &\times \Big(\gamma_\mu (i\not{p}' - i\not{q}_1 - m)\gamma_\nu (i\not{p}' - i\not{q}_1 - i\not{q}_2 - m)\gamma_\lambda\Big) \\ &\times \frac{\partial}{\partial k_\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2).\end{aligned}$$

Transition to a Euclidean coordinate-space representation

- Interchange the integrals over momenta and positions
- Write the momenta $\not{q}_1$ and $\not{q}_2$ in the numerator as derivatives with respect to x and y

$$\Gamma_{\rho\sigma}(p,p) \;=\; -e^6 \int_{x,y} K_{\mu\nu\lambda}(p,x,y)\, \widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y),$$

with the QED kernel

$$\begin{aligned} K_{\mu\nu\lambda}(p,x,y) \quad &= \quad \gamma_\mu(i\not{p}+\not{\partial}^{(x)}-m)\gamma_\nu(i\not{p}+\not{\partial}^{(x)}+\not{\partial}^{(y)}-m)\gamma_\lambda\, \mathcal{I}(p,x,y)_{\text{IR reg.}}\,, \\ \mathcal{I}(p,x,y)_{\text{IR reg.}} \quad &= \quad \int_{q,k} \frac{1}{q^2\, k^2\, (q+k)^2}\, \frac{1}{(p-q)^2+m^2}\, \frac{1}{(p-q-k)^2+m^2}\; e^{-i(q\cdot x+k\cdot y)}\,. \end{aligned}$$

and

$$\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) \;=\; \int_z iz_\rho \,\Big\langle j_\mu(x)j_\nu(y)j_\sigma(z)j_\lambda(0)\Big\rangle_{\text{QCD}}$$

An infrared divergence in the scalar function $\mathcal{I}$ cancels out upon evaluating the Dirac trace and the derivatives.

Simplifying the trace. . .

$$a_\mu^{\rm HLbL} = \frac{me^6}{3}\int_{x,y} \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(p,x,y)\; i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y),$$

with the QED kernel given by

$$\begin{aligned}
&\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(p,x,y)\\
&= \frac{1}{16m^2}\,{\rm Tr}\Big\{(-i\not p+m)[\gamma_\rho,\gamma_\sigma](-i\not p+m)K_{\mu\nu\lambda}(p,x,y)\Big\}\\
&= -\frac{i}{8m}\,{\rm Tr}\Big\{\Big(\not p[\gamma_\rho,\gamma_\sigma]+2(p_\sigma\gamma_\rho-p_\rho\gamma_\sigma)\Big)\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\lambda\Big\}\,\partial_\alpha^{(x)}(\partial_\beta^{(x)}+\partial_\beta^{(y)})\,\mathcal{I}\\
&+\frac{1}{4m}\,{\rm Tr}\Big\{\Big(\not p[\gamma_\rho,\gamma_\sigma]+2(p_\sigma\gamma_\rho-p_\rho\gamma_\sigma)\Big)\gamma_\mu\gamma_\alpha\gamma_\nu\Big\}\;p_\lambda\;\partial_\alpha^{(x)}\,\mathcal{I}\\
&+\frac{1}{4m}\,{\rm Tr}\Big\{\Big(\not p[\gamma_\rho,\gamma_\sigma]+2(p_\sigma\gamma_\rho-p_\rho\gamma_\sigma)\Big)\gamma_\mu(p_\lambda\gamma_\nu\gamma_\beta-p_\beta\gamma_\nu\gamma_\lambda+p_\nu\gamma_\beta\gamma_\lambda)\Big\}\;(\partial_\beta^{(x)}+\partial_\beta^{(y)})\,\mathcal{I}.
\end{aligned}$$

But what about p, the muon momentum, in Euclidean space?

Realizing an on-shell muon in Euclidean

- Any real Euclidean momentum has a non-negative Euclidean norm, $p^2 \geq 0$.
- The pole in p_0 corresponding to on-shell particle with spatial momentum $\boldsymbol{p}$ and mass m lies at $p_0 = \pm i E_{\boldsymbol{p}}$, $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$, for instance

$$\int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \frac{e^{ip_0 x_0}}{p_0^2 + \boldsymbol{p}^2 + m^2} = \frac{e^{-E_{\boldsymbol{p}}|x_0|}}{2E_{\boldsymbol{p}}}.$$

- Thus setting $p = (iE_{\boldsymbol{p}}, \boldsymbol{p})$ puts the muon on its mass shell.
- Simplest choice: $\boldsymbol{p} = 0$ and $p = im\hat{e}_0$. "Imaginary momentum in the time direction".
- At this point, choice of the "time" direction is arbitrary, so can choose $p = im\hat{\epsilon}$, $\hat{\epsilon}^2 = 1$, and average over the direction of $\hat{\epsilon}$ under SO(4).

Averaging over the direction of the muon momentum

We arrive at

$$a_\mu^{\rm HLbL} = \frac{me^6}{3}\int_{x,y} \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)\; i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y),$$

with

$$\begin{aligned}\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \;=\;& \mathcal{G}^{\rm I}_{\delta[\rho,\sigma]\mu\alpha\nu\beta\lambda}\,\langle\hat\epsilon_\delta\partial^{(x)}_\alpha(\partial^{(x)}_\beta+\partial^{(y)}_\beta)\mathcal{I}\rangle_{\hat\epsilon}\\ &+m\,\mathcal{G}^{\rm II}_{\delta[\rho,\sigma]\mu\alpha\nu\beta\lambda}\langle\hat\epsilon_\delta\hat\epsilon_\beta\;\partial^{(x)}_\alpha\mathcal{I}\rangle_{\hat\epsilon}\\ &+m\,\mathcal{G}^{\rm III}_{\delta[\rho,\sigma]\mu\alpha\nu\beta\lambda}\;\langle\hat\epsilon_\alpha\hat\epsilon_\delta(\partial^{(x)}_\beta+\partial^{(y)}_\beta)\mathcal{I}\rangle_{\hat\epsilon},\end{aligned}$$

where we have defined

$$\begin{aligned}\mathcal{G}^{\rm I}_{\delta[\rho,\sigma]\mu\alpha\nu\beta\lambda} &\equiv \frac{1}{8}{\rm Tr}\Big\{\Big(\gamma_\delta[\gamma_\rho,\gamma_\sigma]+2(\delta_{\delta\sigma}\gamma_\rho-\delta_{\delta\rho}\gamma_\sigma)\Big)\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\lambda\Big\},\\ \mathcal{G}^{\rm II}_{\delta[\rho,\sigma]\mu\alpha\nu\beta\lambda} &\equiv -\frac{1}{4}{\rm Tr}\Big\{\Big(\gamma_\delta[\gamma_\rho,\gamma_\sigma]+2(\delta_{\delta\sigma}\gamma_\rho-\delta_{\delta\rho}\gamma_\sigma)\Big)\gamma_\mu\gamma_\alpha\gamma_\nu\Big\}\;\delta_{\beta\lambda},\\ \mathcal{G}^{\rm III}_{\delta[\rho,\sigma]\mu\alpha\nu\beta\lambda} &\equiv -\frac{1}{4}{\rm Tr}\Big\{\Big(\gamma_\delta[\gamma_\rho,\gamma_\sigma]+2(\delta_{\delta\sigma}\gamma_\rho-\delta_{\delta\rho}\gamma_\sigma)\Big)\gamma_\mu(\delta_{\alpha\lambda}\gamma_\nu\gamma_\beta-\delta_{\alpha\beta}\gamma_\nu\gamma_\lambda+\delta_{\alpha\nu}\gamma_\beta\gamma_\lambda)\Big\}.\end{aligned}$$

The tensors $\mathcal{G}^{\rm A}_{\delta[\rho,\sigma]\mu\alpha\nu\beta\lambda}$ are sums of products of Kronecker deltas.

Tensor decomposition

Define

$$
\begin{aligned}
S(x,y) &= \Big\langle \mathcal{I}(p,x,y)_{\text{IR reg.}} \Big\rangle_{\hat\epsilon}, \\
V_\delta(x,y) &= \Big\langle \hat\epsilon_\delta\, \mathcal{I}(p,x,y)_{\text{IR reg.}} \Big\rangle_{\hat\epsilon}, \\
T_{\beta\delta}(x,y) &= \Big\langle \Big(\hat\epsilon_\beta \hat\epsilon_\delta - \frac{1}{4}\delta_{\beta\delta}\Big)\, \mathcal{I}(p,x,y)_{\text{IR reg.}} \Big\rangle_{\hat\epsilon}.
\end{aligned}
$$

Then the QED kernel $\bar{\mathcal{L}}$ can be expressed as linear combinations with integer coefficients of the rank-3 tensors

$$
\begin{aligned}
T^{\rm I}_{\alpha\beta\delta}(x,y) &= \partial^{(x)}_\alpha (\partial^{(x)}_\beta + \partial^{(y)}_\beta) V_\delta(x,y), \\
T^{\rm II}_{\alpha\beta\delta}(x,y) &= m\partial^{(x)}_\alpha \Big(T_{\beta\delta}(x,y) + \frac{1}{4}\delta_{\beta\delta} S(x,y) \Big), \\
T^{\rm III}_{\alpha\beta\delta}(x,y) &= m(\partial^{(x)}_\beta + \partial^{(y)}_\beta) \Big(T_{\alpha\delta}(x,y) + \frac{1}{4}\delta_{\alpha\delta} S(x,y) \Big).
\end{aligned}
$$

The tensor $S, V_\delta, T_{\beta\delta}$ correspond to the scalar, vector and rank-2 tensor components of the function $\mathcal{I}$ with respect to its dependence on $\hat\epsilon$.

The scalar function $\mathcal{I}$

Recall:

$$\mathcal{I}(p,x,y)_{\rm IR\ reg.} \quad = \quad \int_{q,k} \frac{1}{q^2\, k^2\, (q+k)^2}\, \frac{1}{(p-q)^2+m^2}\, \frac{1}{(p-q-k)^2+m^2}\; e^{-i(q\cdot x+k\cdot y)}\,.$$

In terms of position-space propagators, we can write it as

$$\begin{aligned} \mathcal{I}(p=im\hat\epsilon,x,y) &= \int_u G_0(y-u)\,J(\hat\epsilon,u)\,J(\hat\epsilon,x-u)\,, \\ J(\hat\epsilon,u) &= \int_{\tilde u} G_0(u-\tilde u)e^{m\hat\epsilon\cdot\tilde u}\,G_m(\tilde u)\,. \end{aligned}$$

The function $J(\hat\epsilon,u)$ represents the amplitude for a scalar particle to start from the origin, emit a photon that reaches spacetime-point u, and emerge on-shell.

Propagators in Euclidean:

$$\begin{aligned} G_0(x-y) &= \int_k \frac{e^{ik\cdot(x-y)}}{k^2} = \frac{1}{4\pi^2(x-y)^2}\,, \\ G_m(x-y) &= \int_k \frac{e^{ik\cdot(x-y)}}{k^2+m^2} = \frac{m}{4\pi^2|x-y|}K_1(m|x-y|)\,, \end{aligned}$$

The function $J(\hat\epsilon, u)$

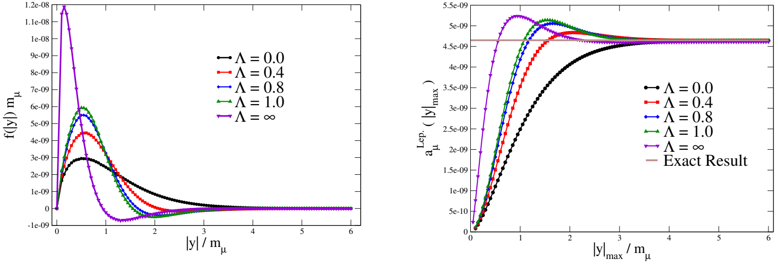
Its expansion in $\lambda = 1$ Gegenbauer polynomials
(analogue for $d = 4$ of Legendre polynomials for $d = 3$):

$$
\begin{aligned}
J(\hat\epsilon, u) &= \frac{1}{8\pi^2 m|u|} \int_0^{m|u|} dt\, e^{t\hat\epsilon\cdot\hat u}\, K_0(t) = \sum_{n=0}^{\infty} z_n(u^2)\, C_n(\hat\epsilon\cdot\hat u),\\
z_n(u^2) &= \frac{1}{4\pi^2}\Big[I_{n+2}(m|u|)\frac{K_0(m|u|)}{n+1} + I_{n+1}(m|u|)\Big(\frac{K_1(m|u|)}{n+1} + \frac{K_0(m|u|)}{m|u|}\Big)\Big],
\end{aligned}
$$

The average of the scalar, vector, tensor components of $J(\hat\epsilon, u)\, J(\hat\epsilon, x-u)$ over $\hat\epsilon$ is done analytically *before* the u integral.

The final u integral is reduced to one angular, one radial integral, which were done numerically.

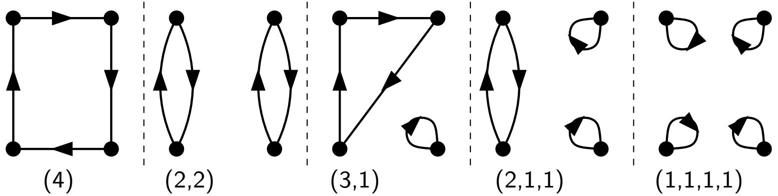
Tests of the framework and adjustments to the kernel



Integrands (Lepton loop, method 2) — Correspondings integrals

- The QED kernel $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ is parametrized by six 'weight' functions of the variables $(x^2, x\cdot y, y^2)$.
- $$\bar{\mathcal{L}}^{(\Lambda)}_{[\rho,\sigma];\mu\nu\lambda}(x,y) = \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) - \partial^{(x)}_\mu (x_\alpha e^{-\Lambda m_\mu^2 x^2/2}) \bar{\mathcal{L}}_{[\rho,\sigma];\alpha\nu\lambda}(0,y) - \partial^{(y)}_\nu (y_\alpha e^{-\Lambda m_\mu^2 y^2/2}) \bar{\mathcal{L}}_{[\rho,\sigma];\mu\alpha\lambda}(x,0),$$
- Using this kernel, we have reproduced (at the 1% level) known results for a range of masses for:
  1. the lepton loop (spinor QED, shown in the two plots);
  2. the charged pion loop (scalar QED);
  3. the π^0 exchange with a VMD-parametrized transition form factor.

Wick-contraction topologies in HLbL amplitude $\langle 0|T\{j_x^\mu j_y^\nu j_z^\lambda j_0^\sigma\}|0\rangle$



First two classes of diagrams turn out to be dominant, with a cancellation between them.

Example: $\Pi = \langle (j_u - j_d)(j_u - j_d)(j_u - j_d)(j_u - j_d)\rangle$ does not contain the π^0 pole (π^0 only couples to one isovector, one isoscalar current).

Write out the Wick contractions: $\Pi = 2\cdot\Pi^{(4)} + 4\cdot\Pi^{(2,2)}$

In kinematic regime where π^0 dominates: $|\Pi| \ll \Pi^{(4)} \Rightarrow \Pi^{(2,2)} \approx -\frac{1}{2}\Pi^{(4)}$.

Including charge factors: $\Big[(Q_u^2+Q_d^2)^2\Pi^{(2,2)}\Big] = -\frac{25}{34}\Big[(Q_u^4+Q_d^4)\Pi^{(4)}\Big]$.

Large-N_c argument by J. Bijnens, 1608.01454; see also 1712.00421.

Rearrangement of integrals: 'method 2'

For the fully-connected calculation we use the following master equation for the integrand:

$$f^{(\text{Conn.})}(|y|) = -\sum_{j\in u,d,s} \hat{Z}_{\rm V}^4 Q_j^4 \frac{m_\mu e^6}{3} 2\pi^2 |y|^3 \times$$

$$\int_x \left(\mathcal{L}'_{[\rho,\sigma]\mu\nu\lambda}(x,y) \int_z z_\rho \widetilde{\Pi}^{(1),j}_{\mu\nu\sigma\lambda}(x,y,z) + \bar{\mathcal{L}}^{(\Lambda)}_{[\rho,\sigma];\lambda\nu\mu}(x,x-y) x_\rho \int_z \widetilde{\Pi}^{(1),j}_{\mu\nu\sigma\lambda}(x,y,z) \right),$$

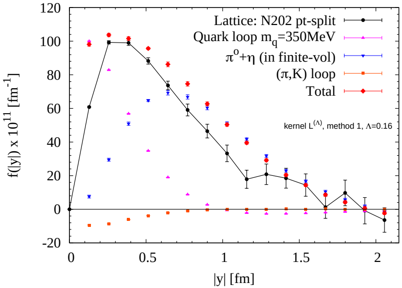
with hadronic contribution

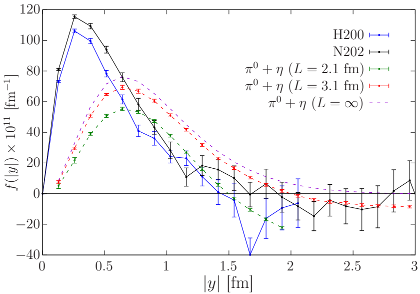
$$\widetilde{\Pi}^{(1),j}_{\mu\nu\sigma\lambda}(x,y,z) = -2\text{Re}\left\langle \text{Tr}\left[S^j(0,x)\gamma_\mu S^j(x,y)\gamma_\nu S^j(y,z)\gamma_\sigma S^j(z,0)\gamma_\lambda \right] \right\rangle_U .$$

- $S^j(x,y)$ is the flavour j-quark propagator from source y to sink x;
- Q_j is the charge factor ($Q_u = \frac{2}{3}$, $Q_d = -\frac{1}{3}$, $Q_s = -\frac{1}{3}$);
- $\langle\cdot\rangle_U$ denotes the ensemble average.

$$\mathcal{L}'_{[\rho,\sigma];\mu\nu\lambda}(x,y) = \bar{\mathcal{L}}^{(\Lambda)}_{[\rho,\sigma];\mu\nu\lambda}(x,y) + \bar{\mathcal{L}}^{(\Lambda)}_{[\rho,\sigma];\nu\mu\lambda}(y,x) - \bar{\mathcal{L}}^{(\Lambda)}_{[\rho,\sigma];\lambda\nu\mu}(x,x-y).$$

Integrand at $m_\pi = m_K \simeq 415$ MeV



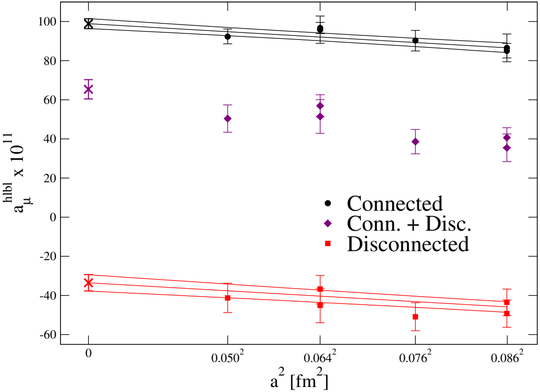


- Partial success in understanding the integrand in terms of familiar hadronic contributions.
- Reasonable understanding of magnitude of finite-size effects. ($L_{\rm H200} = 2.1$ fm, $L_{\rm N202} = 3.1$ fm)

2006.16224 Chao et al. (EPJC)

$a_\mu^{\rm HLbL}$ at $m_\pi = m_K \simeq 415$ MeV: continuum limit

[Chao, Gérardin, Green, Hudspith, HM 2006.16224 (EPJC)]



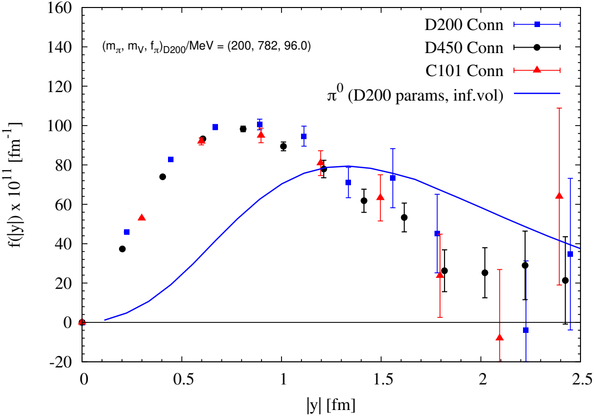
$$a_\mu^{\rm hlbl, SU(3)_f} = (65.4 \pm 4.9 \pm 6.6) \times 10^{-11}.$$

$N_f = 2 + 1$ **CLS ensembles used towards physical quark masses**

	(4)	(22)	(31)	(211)	(1111)	β	$(a\ \text{GeV})^2$	$(\frac{m_\pi}{\text{GeV}})^2$	$(\frac{m_K}{\text{GeV}})^2$	$m_\pi L$	$\hat{Z}_V$
A653	l, s	l, s	0	0	0	3.34	**0.2532**	**0.171**	**0.171**	**5.31**	**0.70351**
A654	l, s	l, s	l				**0.2532**	**0.107**	**0.204**	**4.03**	**0.69789**
U103	l, s	l, s	0	0	0		0.1915	0.172	0.172	4.35	0.71562
H101	l, s	l, s	0	0	0		0.1915	0.173	0.173	5.82	0.71562
U102	l	l	l			3.40	0.1915	0.127	**0.194**	3.74	0.71226
H105	l, s	l, s	l, s				0.1915	0.0782	0.213	3.92	0.70908
C101	l, s	l, s	l, s	l	l, s		0.1915	0.0488	0.237	4.64	0.70717
B450	l, s	l, s	0	0	0	3.46	0.1497	0.173	0.173	5.15	0.72647
D450	l	l	l				0.1497	**0.0465**	0.226	**5.38**	**0.71921**
H200	l, s	l, s	0	0	0		0.1061	0.175	0.175	4.36	0.74028
N202	l, s	l, s	0	0	0		0.1061	0.168	0.168	6.41	0.74028
N203			l	l		3.55	0.1061	0.120	0.194	5.40	0.73792
N200	l	l	l				0.1061	0.0798	0.214	4.42	0.73614
D200	l	l	l				0.1061	0.0397	0.230	4.15	0.73429
N300	l, s	l, s	0	0	0	3.70	0.06372	0.178	0.178	5.11	0.75909

En-Hung Chao, Renwick Hudspith, Antoine Gérardin, Jeremy Green, HM, Konstantin Ottnad
2104.02632 (EPJC)

Integrand of connected contribution at $m_\pi \approx 200$ MeV

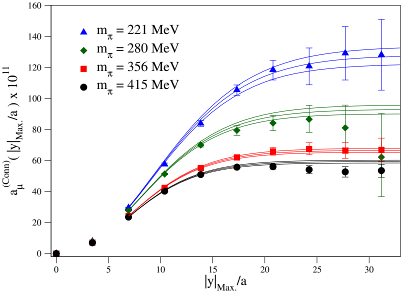


- using four local vector currents
- based on 'Method 2' with improved kernel $\bar{\mathcal{L}}^{(\Lambda)}$.

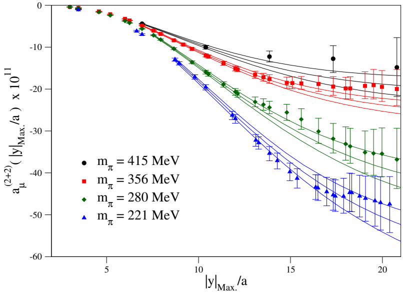
2104.02632

Truncated integral for a_μ^{HLbL}

Connected



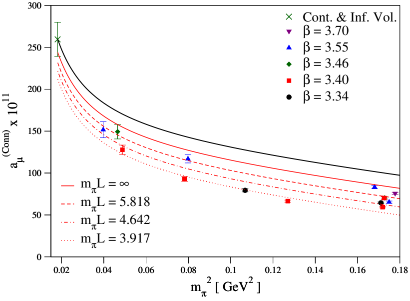
(2+2) Disconnected



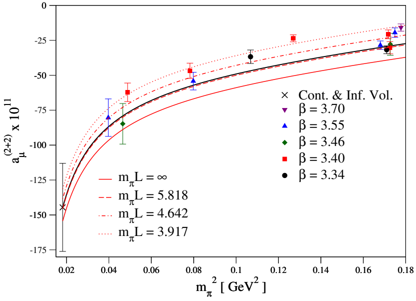
- Extend reach of the signal by two-param. fit $f(y) = A|y|^3 \exp(-M|y|)$;
- provides an excellent description of the π^0 exchange contribution in infinite volume.
- We see a clear increase of the magnitude of both connected and disconnected contributions.

Chiral, continuum, volume extrapolation

Connected contribution

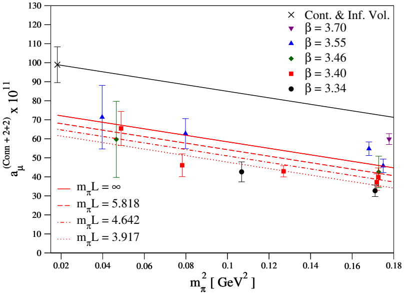


disconnected contribution



Total light-quark contribution:

- vol. dependence: $\propto \exp(-m_\pi L/2)$
- pion-mass dependence fairly mild (!)



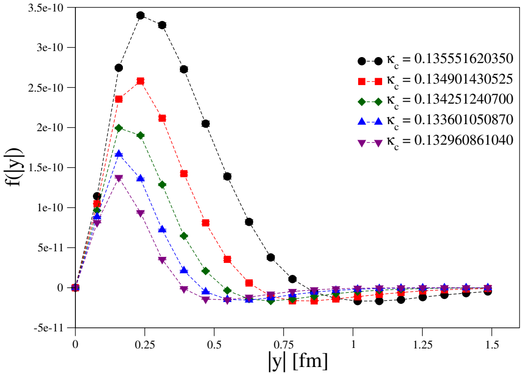
Overview table

Contribution	Value$\times 10^{11}$
Light-quark fully-connected and $(2+2)$	107.4(11.3)(9.2)(6.0)
Strange-quark fully-connected and $(2+2)$	$-0.6(2.0)$
$(3+1)$	0.0(0.6)
$(2+1+1)$	0.0(0.3)
$(1+1+1+1)$	0.0(0.1)
Total	106.8(15.9)

- error dominated by the statistical error and the continuum limit.
- all subleading contributions have been tightly constrained and shown to be negligible.

[Chao et al, 2104.02632]

The charm contribution at the SU(3)$_f$ point

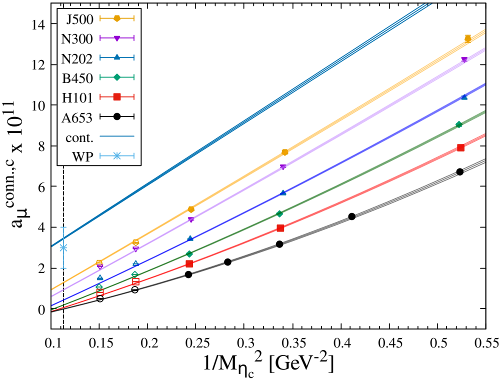


Integrand for the connected charm contribution (J500, $a = 0.039\,\text{fm}$)

- direct calculation at physical charm mass difficult due to lattice artefacts
- $\rightsquigarrow$ perform a combined extrapolation in $1/m_c^2$ and the lattice spacing.

Chao, Hudspith, Gérardin, Green, HM arXiv:2204.08844

Extrapolation in charm mass and lattice spacing

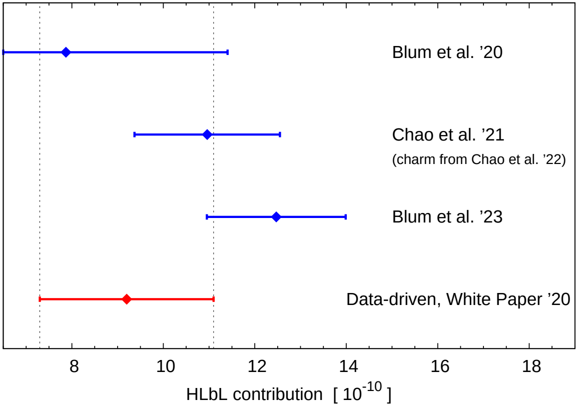


This particular fit:

$$a_\mu(a, m_{\eta_c}) = Aa + \frac{B + Ca^2}{m_{\eta_c}^2} + Da^2 + E\frac{a^2}{m_{\eta_c}^4}$$

Final result (average of several fits): $a_\mu(\text{charm}) = (2.8 \pm 0.5) \times 10^{-11}$.

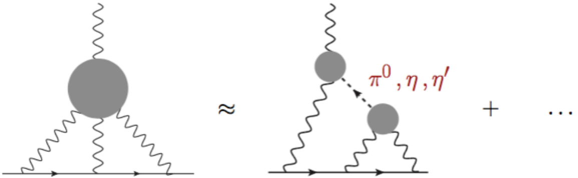
Compilation of $a_\mu^{\rm HLbL}$ determinations



Good consistency of different determinations.
Lattice'24: $a_\mu^{\rm HLbL} = 12.6(1.2)(3) \cdot 10^{-10}$ (Ch. Zimmermann, BMW).

Transition form factors of π^0, η, η'

The $\pi^0 \to \gamma^*\gamma^*$ transition form factor from the lattice



$$M_{\mu\nu}(p,q_1) \equiv i \int \mathrm{d}^4x\, e^{iq_1 x}\, \langle \Omega | T\{J_\mu(x) J_\nu(0)\} | \pi^0(p) \rangle = \epsilon_{\mu\nu\alpha\beta}\, q_1^\alpha\, q_2^\beta\, F_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2)\,,$$

Lattice: $M_{\mu\nu}(p,q_1) = (i^{n_0}) M^{\mathrm{E}}_{\mu\nu}(p,q_1)$, where

$$M^{\mathrm{E}}_{\mu\nu}(p,(\omega_1,\boldsymbol{q}_1)) = \frac{2E_\pi}{Z_\pi} \int_{-\infty}^{\infty} \mathrm{d}\tau\, e^{\omega_1 \tau}\, \widetilde{A}_{\mu\nu}(\tau),$$

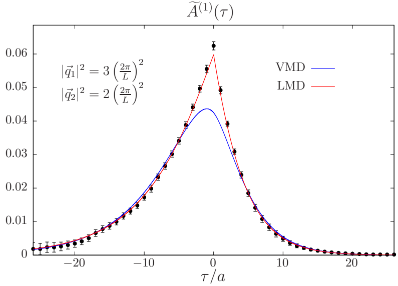
$$\widetilde{A}_{\mu\nu}(\tau) = \lim_{t_\pi \to +\infty} e^{E_{\boldsymbol{p}}(t_f - t_0)} C^{(3)}_{\mu\nu}(\tau, t_\pi),$$

$$C^{(3)}_{\mu\nu}(\tau, t_\pi) = a^6 \sum_{\boldsymbol{x},\boldsymbol{z}} \left\langle J_\mu(\boldsymbol{z}, t_i) J_\nu(\boldsymbol{0}, t_f) P^\dagger(\boldsymbol{x}, t_0) \right\rangle e^{i\boldsymbol{p}\boldsymbol{x}}\, e^{-i\boldsymbol{q}_1 \boldsymbol{z}}\,.$$

$\tau = t_i - t_f$, $t_\pi = \min(t_f - t_0, t_i - t_0)$.

Some early Refs: Ji, Jung hep-lat/0101014; Dudek, Edwards, hep-ph/0607140; Feng et al. 1206.1375; A. Gérardin, HM, A. Nyffeler 1607.08174.

Inverse relation: the amplitude $\widetilde{A}_{\mu\nu}$ from the transition form factor



$$\widetilde{A}_{0k}(\tau) = (\boldsymbol{q}_1 \times \boldsymbol{p})^k \, \widetilde{A}^{(1)}(\tau),$$

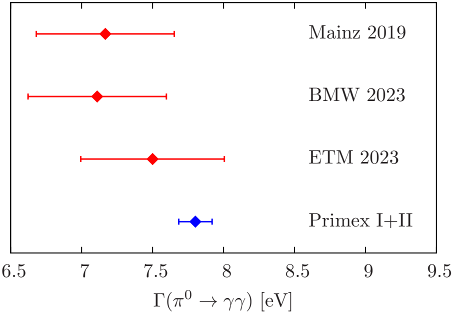
$$\epsilon'^k \, \widetilde{A}_{kl}(\tau) \, \epsilon^l = -i(\boldsymbol{\epsilon}' \times \boldsymbol{\epsilon}) \cdot \left(\boldsymbol{q}_1 \, E_{\boldsymbol{p}} \, \widetilde{A}^{(1)}(\tau) + \boldsymbol{p} \, \frac{d\widetilde{A}^{(1)}}{d\tau}\right).$$

where

$$\widetilde{A}^{(1)}(\tau) = \frac{iZ_\pi}{4\pi E_\pi} \int_{-\infty}^{\infty} \mathrm{d}\tilde{\omega} \, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) e^{-i\tilde{\omega}\tau} \,,$$

$$q_1^2 = -(\tilde{\omega}^2 + \boldsymbol{q}_1^2)$$
$$q_2^2 = (E_\pi - i\tilde{\omega})^2 - (\boldsymbol{p} - \boldsymbol{q}_1)^2.$$

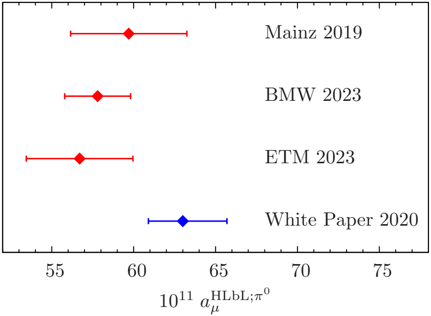
Recent results: pion lifetime



$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{\pi\alpha^2 m_{\pi^0}^3}{4} \mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)^2$$

Gérardin, HM, Nyffeler 1903.09471; BMW 2305.04570; ETMC 2308.12458.

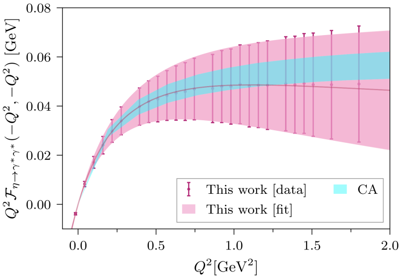
Recent results: contribution to a_μ^{HLbL}

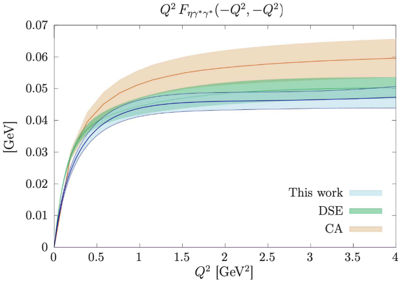


$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau$$
$$\Big[w_1(Q_1,Q_2,\tau)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-(Q_1+Q_2)^2)\ \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2,0)$$
$$+w_2(Q_1,Q_2,\tau)\ \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)\ \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2,0)\Big].$$

Gérardin, HM, Nyffeler 1903.09471 (PRD); BMW 2305.04570; ETMC 2308.12458 (PRD).
For the expression of $a_\mu^{\mathrm{HLbL};\pi^0}$, see Nyffeler 1602.03398.

The η transition form factor



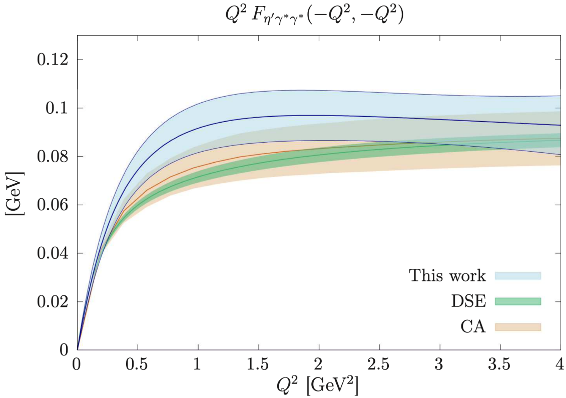


$$\Gamma(\eta\to\gamma\gamma)[\text{eV}] = \begin{cases} 338(87)_{\text{stat}}(17)_{\text{syst}} & \text{ETM22} \\ 338(94)_{\text{stat}}(35)_{\text{syst}} & \text{BMW23} \end{cases} \qquad \text{PDG}: 516(18).$$

$$10^{11}a_\mu^{\text{HLbL};\eta} = \begin{cases} 13.8(5.2)_{\text{stat}}(1.5)_{\text{syst}} & \text{ETM22} & \text{WP20}: 16.3(1.4) \\ 11.6(1.6)_{\text{stat}}(0.5)_{\text{syst}}(1.1)_{\text{FSE}} & \text{BMW23} & \text{(Canterbury approx)} \end{cases}$$

Left: ETMC 2212.06704 (PRD). Right: BMW 2305.04570.

The η' transition form factor



$$\Gamma(\eta' \to \gamma\gamma)[\text{keV}] = 3.4(1.0)_{\text{stat}}(0.4)_{\text{syst}} \qquad \text{PDG} : 4.28(19).$$

$$10^{11} a_\mu^{\text{HLbL};\eta'} = 15.7(3.9)_{\text{stat}}(1.1)_{\text{syst}}(1.3)_{\text{FSE}} \qquad \text{WP20} : 14.5(1.9)$$

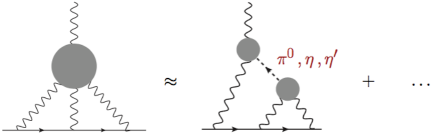
(Canterbury approx)

BMW 2305.04570.

Conclusion on a_μ^{HLbL}

- Results from the Bern dispersive framework and from three independent lattice QCD calculations since 2021 are in agreement with comparable uncertainties.
- All three lattice results lie above the WP20 value.
- Good progress on the pseudoscalar transition form factors, with first calculations available for η and η'.

Models for a_μ^{HLbL}



Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99 ± 16
axial vectors	2.5±1.0	1.7±1.7	–	22±5	–	15 ± 10	22 ± 5
scalars	−6.8±2.0	–	–	–	–	−7±7	−7±2
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
π, K loops +subl. N_C	–	–	–	0±10	–	–	–
quark loops	21±3	9.7±11.1	–	–	–	2.3 (c-quark)	21±3
Total	83±32	89.6±15.4	80±40	136±25	110±40	105 ± 26	116 ± 39

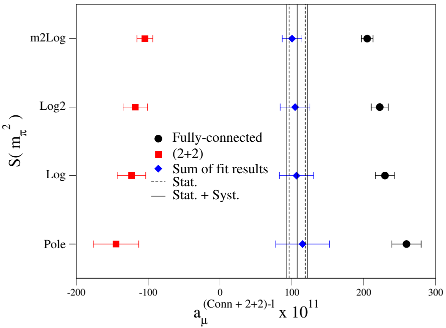
BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = AN '09, JN = Jegerlehner, AN '09

Table from A. Nyffeler, PhiPsi 2017 conference

One further estimate: NB. much smaller axial-vector contribution

$$a_\mu^{\mathrm{HLbL}} = (103 \pm 29) \times 10^{-11} \qquad \text{Jegerlehner 1809.07413}$$

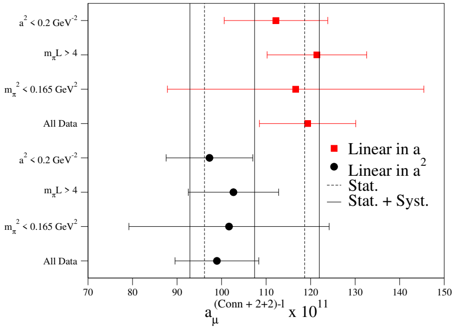
Separate extrapolation of conn. & disconn.



$$\text{Ansatz :} \quad Ae^{-m_\pi L/2} + Ba^2 + CS(m_\pi^2) + D + Em_\pi^2$$

- chirally singular behaviour cancels in sum of connected and disconnected.

Extrapolation to the sum of conn. & disconn.

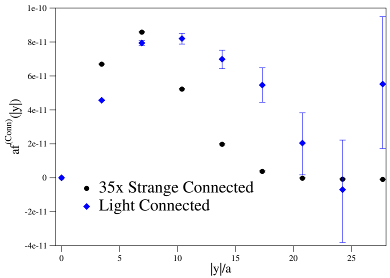


$$\text{Ansatz}: \quad Ae^{-m_\pi L/2} + Ba^2 + D + Em_\pi^2$$

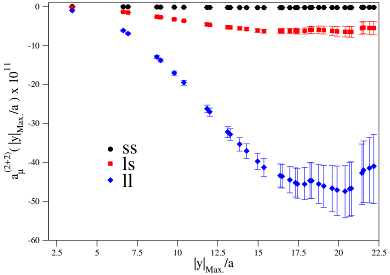
- results very stable with respects to cuts in a, m_π or $m_\pi L$.
- largest systematic comes from choice of continuum limit ansatz.
- final result: central value from fitting these results with a constant; systematic error set to $\sqrt{(1/N)\sum_{i=1}^{N}(y_i - \bar{y})^2}$ as a measure of the spread of the results.

Strange contribution

Ensemble C101 ($48^3 \times 96$, $a = 0.086$ fm, $m_\pi = 220$ MeV)

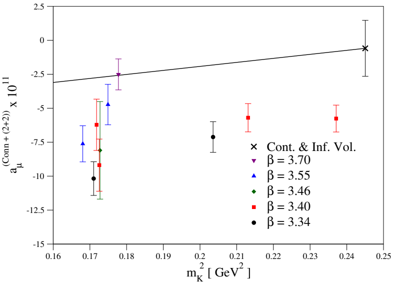


NB. Strange integrand has a factor 17 suppression due to charge factor.



(2,2) disconnected contributions.

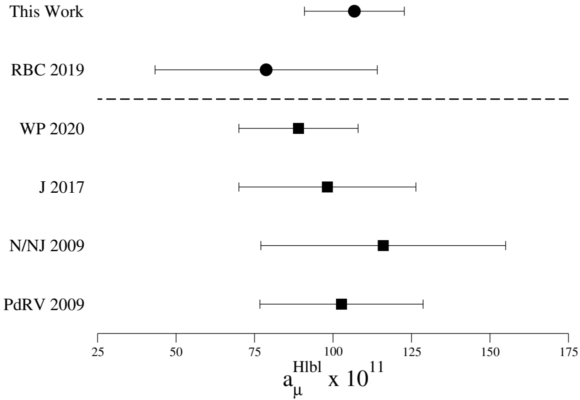
Extrapolation of strange contributions



Sum of connected-strange + (2,2) topology with ss and sl quark-line content.

Final strange contribution is very small as a result of cancellations.

Compilation of a_μ^{HLbL} determinations



Good consistency of different determinations (not including charm here).

Fig from Chao et al, 2104.02632 (EPJC).